Nonzero bound on Fiedler eigenvalue causes exponential growth of H-infinity norm of vehicular platoon

Ivo Herman, Dan Martinec, Zdeněk Hurák, Michael Šebek

Abstract—We consider platoons composed of identical vehicles and controlled in a distributed way, that is, each vehicle has its own onboard controller. The regulation errors in spacing to the immediately preceding and following vehicles are weighted differently by the onboard controller, which thus implements an asymmetric bidirectional control scheme. The weights can vary along the platoon. We prove that such platoons have a nonzero uniform bound on the second smallest eigenvalue of the graph Laplacian matrix—the Fiedler eigenvalue. Furthermore, it is shown that existence of this bound always signals undesirable scaling properties of the platoon. Namely, the $\mathcal{H}_\infty$ norm of the transfer function of the platoon grows exponentially with the number of vehicles regardless of the controllers used. Hence the benefits of a uniform gap in the spectrum of a Laplacian with an asymmetric distributed controller are paid for by poor scaling as the number of vehicles grows.

Index Terms—Vehicular platoons, Fiedler eigenvalue, harmonic instability, eigenvalues uniformly bounded from zero, asymmetric control, exponential scaling.

I. INTRODUCTION

Platoons of vehicles offer promising solutions for future highway transport. They provide several advantages to the current highway traffic—they increase both the capacity and the safety of the highway and they allow the driver to relax.

Several platoon control architectures have been proposed in the literature. They differ mainly in presence of direct interactions with the platoon leader. If the information from the leader is permanently available to all following vehicles, the platoon can behave very well and is scalable. On the other hand, it requires establishing some communication among the vehicles, which can be disturbed or even denied by an intruder. Control schemes relying on communication comprise leader following and cooperative adaptive cruise control. For their overview and properties see e. g., [1]–[3].

Among the communication-free scenarios are the predecessor following, constant time-headway spacing and bidirectional control (symmetric or asymmetric). Recognizing their limitations, these architectures may still be useful as backup control solutions during communication failures. One of the key theoretical issues investigated with communication-free control schemes is string stability. Although there are variations among the concepts found in the literature (for a review see, e.g., [4]), the key idea is that the platoon is string unstable if the impact of a disturbance affecting one vehicle gets amplified as it propagates along the string (platoon). The predecessor-following strategy is string unstable if there are at least two integrators in the open loop of each vehicle [1]. Two integrators are a reasonable assumption, as they allow both velocity tracking and constant spacing [5]. The constant time-headway spacing policy increases the required intervehicular distances in response to the increased speed of the leader, which preserves the string stability [6]. For symmetric bidirectional formations, the response to noise (coherence) scales polynomially with the size of the platoon [7]. The paper also reveals the bad effect of increasing the number of integrators in the open loop.

Asymmetric controllers for platoons have received much attention after [8] was published. The authors show that for small controller asymmetry, the convergence rate of the least stable eigenvalue to zero (as the number of vehicles grows) decreases. Later the paper [9] shows that with a non-vanishingly small asymmetry, the least stable eigenvalue does not actually converge to zero but to some nonzero constant—a uniform nonzero lower bound can be achieved. This result guarantees a controllability of the formation consisting of an arbitrary number of vehicles. In [10] optimal localized control for asymmetric formation is proposed. The authors show that asymmetric control has beneficial effects on various performance measures. They do, however, assume that each vehicle in the platoon has the knowledge of the desired (leader’s) velocity of the platoon. That information has to be communicated permanently to each vehicle by the leader. Our work to be presented differs in that we allow no communication among the vehicles.

The results in [11] reveal a significant drawback of the asymmetric control scheme. The paper analyzes a platoon of vehicles modeled by double integrators with a PD controller (equivalent to relative position and velocity feedback). They show that the peak in the magnitude frequency response of the position of the last vehicle to the change in the leader’s position grows exponentially in the number of vehicles—a phenomenon labelled as harmonic instability. In contrast, if the controller is symmetric, the peak in the magnitude frequency response (the $\mathcal{H}_\infty$ norm system) only grows linearly [12]. Note that string instability merely means that the $\mathcal{H}_\infty$ norm is growing but harmonic instability means that it is growing very fast (in the number of vehicles).

With these results, several questions arise. Is harmonic instability present with any controller or can it be mitigated by some judicious choice of the controller structure? Can varying the asymmetry in the platoon counteract harmonic instability? Is harmonic instability an inherent property of an asymmetric control, or even of any uniformly bounded nearest neighbor interaction? In this paper we answer these questions.

We extend [11] to any open-loop model of a vehicle and any platoon with uniformly bounded eigenvalues. Our results also extend [1] from the predecessor-following architecture to any bidirectional asymmetric configuration. Moreover, we show that harmonic instability is, in fact, caused by the uniform boundedness and it is not possible to achieve a good scalability both in the convergence time (the bound on eigenvalues) and in the frequency domain (the $\mathcal{H}_\infty$ system norm). Some trade-off is necessary. This paper extends our previous conference...
paper [13] to arbitrary asymmetric formations with controller gains and asymmetries varying among the vehicles.

The paper is structured as follows. First we give some preliminaries and provide definition of the harmonic instability. Then we prove uniform boundedness of a general platoon. In the next section the proof for the harmonic instability of an asymmetric control scheme is given. Finally some special cases are discussed and simulation results are shown.

II. PRELIMINARIES AND MODEL

We assume $N$ vehicles indexed by $i = 1, 2, \ldots, N$, travelling in a one-dimensional space. The first vehicle (indexed 1) is called the leader and it is controlled independently of the rest of the platoon. We analyze a bidirectional control, where each onboard controller measures the distances to its immediate predecessor and follower and strives to keep these close to the desired (reference) distance. It sets different weights to the front and rear regulation errors, hence asymmetric bidirectional control. We assume no intervehicular communication; all information is obtained only locally by the onboard sensors.

We study how the disturbance created by unexpected movements of the leader propagates along the platoon towards the final vehicle. Hence, we analyze the properties of the transfer function $T_N(s)$ from the leader’s position to the position of the last vehicle as depicted in Fig. 1. Its frequency response and the way it scales with the number of vehicles $N$ is used to prove harmonic instability for a given configuration.

**Definition 1** (Harmonic stability [11]). Let \( \gamma_N \equiv \sup_{\omega \in \mathbb{R}_+} |T_N(j \omega)| \), where $j = \sqrt{-1}$. The platoon is called harmonically stable if it is asymptotically stable and if \( \limsup_{N \to \infty} \frac{1}{\gamma_N} \leq 1 \). Otherwise it is harmonically unstable.

An interpretation of harmonic instability is that some oscillatory motion of the leader has its amplitude magnified as it is propagated through the platoon and the growth of the magnitude is exponential in $N$ [11]. In order words, the $H_\infty$ norm of $T_N(s)$ grows exponentially with $N$.

**Notation:** We denote matrices by capital letters, vectors by lowercase letters and an element in a matrix $A$ is denoted as $a_{ij}$. We use $s$ as the complex variable in Laplace transform. The $i$th vector in a canonical basis is denoted $e_i \in \mathbb{R}^{N \times 1}$, that is, $e_i = [0, \ldots, 0, 1, 0, \ldots, 0]^T$, with 1 on the $i$th position. Identity matrix of size $N$ is denoted as $I_N$.

**A. System model**

Each vehicle is described by an identical SISO transfer function $G(s) = \frac{k(s)}{\theta(s)}$. The output is the vehicle’s position $y_i$. Dynamic controller described by a transfer function $R(s) = \frac{q(s)}{p(s)}$ is used to close the feedback loop. The input to the controller is defined in (2). The open-loop transfer function is $M(s) = R(s)G(s)$. From now on we will only use the open loop in the analysis. Its state-space description is

\[
\dot{x}_i = A x_i + B u_i, \quad y_i = C x_i,
\]

with $x_i \in \mathbb{R}^{n \times 1}$ as the state vector, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{1 \times n}$, and $y_i \in \mathbb{R}$ as the position of the vehicle.

The input has two parts $u_i = c_i + r_i$, where $c_i$ is the part caused by coupling between vehicles and $r_i$ external control signal, e.g., reference distances.

In platooning, the input $u_i$ to each vehicle is a weighted sum of spacing errors to its predecessor in the string and its successor. Spacing error to the previous vehicle is weighted with $\mu_i > 0$ and the error to succeeding vehicle with $\mu_i \epsilon_i$. The asymmetry level $c_i \geq 0$ is therefore a ratio between front and rear gains. The weight $\mu_i$ as well as the asymmetry level $c_i$ can vary along the platoon. The input to the vehicle is then

\[
u_i = \mu_i (y_{i-1} - y_i - d_{ref}) - \mu_i \epsilon_i (y_i - y_{i+1} - d_{ref}),
\]

for $i = 2, \ldots, N$ and $d_{ref}$ is a desired distance. The intervehicle coupling is then $c_i = \mu_i (y_{i-1} - y_i) - \mu_i \epsilon_i (y_i - y_{i+1})$ and the external input in this case is $r_i = \mu_i (1 + c_i) d_{ref}$. In further development, we do not limit the external command $r_i$ to be given only by the reference distance, it is treated as a general signal.

In a compact form, the stacked vector of inputs is $u = -Ly + r$, where $L$ is a graph Laplacian representing an interconnection and has a form

\[
L = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
-\mu_2 & \mu_2(1 + \epsilon_2) & -\mu_2 \epsilon_2 & 0 & \cdots \\
-\mu_3 & \mu_3(1 + \epsilon_3) & -\mu_3 \epsilon_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & -\mu_N \epsilon_N
\end{bmatrix}.
\]

The vectors are $u = [u_1, \ldots, u_N]^T$, $y = [y_1, \ldots, y_N]^T$ and since the external input can be arbitrary, we take $r = [r_1, \ldots, r_N]^T$. The leader is controlled externally with the input $u_1$. The trailing vehicle has no follower and its input is $u_N = \mu_N (y_{N-1} - y_N - d_{ref})$.

**Lemma 1.** The graph Laplacian in (3) has these properties

a) \( \lambda_1 = 0 \) is an eigenvalue of $L$ with its eigenvector $1 = [1, 1, \ldots, 1]^T$, and this eigenvalue is simple.

b) all its eigenvalues are real, lie in the right-half plane, i.e. $\lambda_i \geq 0$ and are bounded by $\lambda_1 \leq \lambda_{max} = 2 \max(L_{ii})$.

c) $L$ can be partitioned as

\[
L = \begin{bmatrix}
0 & \mathbf{0} \\
\mathbf{0} & L_r
\end{bmatrix}.
\]

and the spectrum of the reduced Laplacian $L_r$ coincides with all the nonzero eigenvalues of $L$.

**Proof.** a) Simple zero eigenvalue follows from the presence of a directed spanning tree in the platoon. b) $L$ is a triadigonal real matrix with non-positive off-diagonal terms, so its eigenvalues are real [14, Lem. 0.1.1]. The upper bound follows
from Geršgorin’s theorem [15, Thm. 6.1.1]. c) Combining the property that the first row of \( L \) is zero and one of the eigenvalues is zero, similarity transformation reveals the eigenstructure described in the lemma. □

Using the last point, we can concentrate on the formation without the leader, because we removed the row corresponding to the leader and still kept all the nonzero eigenvalues. That’s why the input to \( T_N(s) \) acts at the second vehicle.

**Definition 2** (Uniform boundedness). The eigenvalues \( \lambda_i \) of a matrix \( L \in \mathbb{R}^{N \times N} \) are uniformly bounded from zero if there exists a constant \( \lambda_{\min} > 0 \) such that \( \lambda_i \geq \lambda_{\min} \) for \( i = 2, \ldots, N \) and \( \lambda_{\min} \) does not depend on \( N \).

If the onboard controllers of all vehicles are asymmetric and have front gains stronger than the rear ones, then the uniform boundedness can be achieved. The proof of the following theorem is in the Appendix A.

**Theorem 1.** If there is \( \epsilon_{\max} < 1 \) such that \( \epsilon_i \leq \epsilon_{\max} \) for \( \forall i \) and \( \forall N \), then the nonzero eigenvalues of the Laplacian \( L \) given in (3) are uniformly bounded with \( \lambda_{\min} \geq \frac{(1-\epsilon_{\max})^2}{2N\epsilon_{\max}} \).

**B. Vehicle interconnection and diagonalization**

Using a standard consensus or multi-vehicular formation notation [16], the overall formation model is

\[
\dot{x} = \begin{bmatrix} I_N \otimes A - (I_N \otimes BC)(L \otimes I_n) \end{bmatrix} x + (I_N \otimes B)r,
\]

\[
y = (I_N \otimes C)x,
\]

where \( x \in \mathbb{R}^{Nn \times 1} \) is a stacked state vector and \( \otimes \) is the Kronecker product. We apply the approach of Theorem 1 from [16] but use Jordan instead of Schur decomposition. This will block diagonalize the system. The state transformation is \( x = (V \otimes F\mu)x \), where \( J = V^{-1}LV \) is the Jordan form of \( L \). The matrix \( V = [v_1, \ldots, v_N] \) is formed by (generalized) eigenvectors of \( L \) and \( v_{ji} \) is the \( j \)th element of the vector \( v_i \). A block diagonal system is

\[
\dot{x} = \begin{bmatrix} I_N \otimes A - J \otimes BC \end{bmatrix} \dot{x} + (V^{-1} \otimes B)r,
\]

\[
y = (V \otimes C)x.
\]

Consider a Jordan block in the block diagonal matrix (6). If it is of size one, it has the form

\[
\dot{x}_i = [A - \lambda_i BC]x_i + B\epsilon_i^T V^{-1} r_i, \quad \dot{y}_i = Cx_i
\]

This equation can be viewed as an output feedback system with a feedback gain \( \lambda_i \) and output \( y_i \). Its transfer function is

\[
F_i(s) = \frac{M(s)}{1 + \lambda_i M(s)} = \frac{b(s)q(s)}{a(s)p(s) + \lambda_i b(s)q(s)}.
\]

If the Jordan block has a size larger than one, it corresponds to identical blocks connected in series as in Fig. 2.

In the following we assume that all diagonal blocks are asymptotically stable for all \( N \). As all the eigenvalues \( \lambda_i \) are real, design of a stable system is not a difficult task. We can use, e.g., the synchronization region approach [17] or the root-locus-like approach [18].

**III. HARMONIC INSTABILITY**

To test for harmonic instability, we examine the transfer function \( T_N(s) \). The input to the platoon for such transfer function is \( r = [0, r_2(s), 0, \ldots, 0]^T = e_2r_2(s) \). Based on (8), the input to the diagonal block \( F_i(s) \) is the \( i \)th entry in the vector \( r \) given by \( \dot{r}_i = (V^{-1}) e_2 r_2(s) = gr_2(s) \) with \( g = V^{-1} e_2 \). The output of each block is (see Fig. 2)

\[
\hat{y}_i(s) = F_i(s) \hat{r}_i(s) = F_i(s) g r_2(s).
\]

The position \( y_N \) of the \( N \)th vehicle can be calculated from (7). It is a weighted sum of the outputs of the blocks \( \hat{y} \) with the weights equal to the \( N \)th terms in the eigenvectors \( v_i \)

\[
y_N(s) = \sum_{i=1}^{N} v_{Ni} \hat{y}_i(s) = \left[ \sum_{i=1}^{N} v_{Ni} F_i(s) g r_2(s) \right] r_2(s),
\]

with which we define \( T_N(s) = \frac{y_N(s)}{r_2(s)} = \sum_{i=1}^{N} v_{Ni} F_i(s) g \).

The following product form of \( T_N(s) \) holds for general platoons (both symmetric and asymmetric).

**Theorem 2.** The transfer function from the input of the second vehicle to the position of the last vehicle in the system (5) with Laplacian (3) is given as

\[
T_N(s) = \frac{1}{\mu_2} \prod_{i=2}^{N} \lambda_i F_i(s) = \frac{1}{\mu_2} \prod_{i=2}^{N} T_{\lambda_i}(s),
\]

We introduced \( T_{\lambda_i}(s) = \lambda_i F_i(s) \) as a closed-loop transfer function with gain \( \lambda_i \). The proof is given in the Appendix B.

**Corollary 1.** For at least one integrator in the open loop \( M(s) \), the steady-state gain of each block in (12) is \( T_{\lambda_i}(0) = 1 \), \( \forall i \) and then \( T_N(0) = 1 \) if and only if \( \mu_2 = 1 \).

Surprisingly, the greater the gain \( \mu_2 \) (coupling with the leader), the lower the steady-state gain of the platoon.

Before stating the main theorem of the paper, we need to introduce the notation \( M_{\min}(s) = \lambda_{\min} M(s) \) and define the closed-loop block for such open loop as

\[
T_{\lambda_{\min}}(s) = \frac{M_{\min}(s)}{1 + M_{\min}(s)} = \frac{\lambda_{\min} b(s)q(s)}{a(s)p(s) + \lambda_{\min} b(s)q(s)}.
\]

**Theorem 3.** If the nonzero eigenvalues of Laplacian in (3) are uniformly bounded and \( ||T_{\lambda_{\min}}(s)||_\infty > 1 \), then the platoon is harmonically unstable.
Proof. The condition states that the closed-loop block \( T_{\lambda_{\text{min}}}(s) \) defined in (13) corresponding to the lower bound on the eigenvalues of Laplacian is greater in the \( H_\infty \) norm than one. Let \( \omega_0 \) be the frequency at which the magnitude frequency response of this block attains its maximum. Further let \( \alpha + j\beta \) (with \( \sqrt{-1} = j \)) be the value of the frequency response of the scaled open loop \( M_{\text{min}} \) at \( \omega_0 \), i.e., \( M_{\text{min}}(\omega_0) = \alpha + j\beta \). Then the squared modulus of the frequency response of the closed-loop \( T_{\lambda_{\text{min}}}(s) \) reads

\[
|T_{\lambda_{\text{min}}}(j\omega_0)|^2 = \left[ \frac{M_{\text{min}}(j\omega_0)}{1 + M_{\text{min}}(j\omega_0)} \right]^2 = \frac{\alpha^2 + \beta^2}{(\alpha + j\beta)^2 + \beta^2}.
\]

(14)

Since at \( \omega_0 \) the closed-loop magnitude frequency response attains its maximum, the peak is greater than 1, i.e., \( |T_{\lambda_{\text{min}}}(0)| = 1 < |T_{\lambda_{\text{min}}}(j\omega_0)| \). From (14) we have

\[
\frac{\alpha^2 + \beta^2}{(\alpha + j\beta)^2 + \beta^2} > 1 \Rightarrow \alpha < -\frac{1}{2}.
\]

(15)

The Laplacian eigenvalues can be ordered as \( \lambda_2 < \ldots < \lambda_{\text{min}} \leq \lambda_{\text{max}} \). We can write \( \lambda_i = \kappa_i \lambda_{\text{min}} \) with \( \kappa_i = \frac{\lambda_i}{\lambda_{\text{min}}} \). By Lemma 1 all eigenvalues are real, so \( \kappa \) is real as well. By assumption in the theorem the bounds on \( \kappa \) do not depend on the number of vehicles.

Now the transfer function of each term in the product (12) is

\[
T_{\lambda_i}(s) = \frac{\kappa_i M_{\text{min}}(s)}{1 + \kappa_i M_{\text{min}}(s)}
\]

with \( \kappa_i = \frac{\lambda_i}{\lambda_{\text{min}}} \). We prove that all such transfer functions also have the magnitude frequency response at \( \omega_0 \) greater than 1 (not necessarily their maximum there). The value of \( M_{\text{min}}(j\omega_0) \) is still written as \( \alpha + j\beta \). The squared modulus of the closed-loop frequency response at \( \omega_0 \) is

\[
|T_{\lambda_i}(j\omega_0)|^2 = \left[ \frac{\kappa_i M_{\text{min}}(j\omega_0)}{1 + \kappa_i M_{\text{min}}(j\omega_0)} \right]^2 = 1 - \frac{2\kappa_i \alpha + 1}{(\kappa_i + 1)^2 + \kappa_i^2 \beta^2}.
\]

(16)

Since \( \alpha < -\frac{1}{2} \), \( \kappa_i \) is real and greater than 1 and the denominator is positive, the sign of the fraction must be negative and (16) is greater than 1. Therefore, all transfer functions \( T_{\lambda_i}(s) \) at \( \omega_0 \) have the modulus greater than 1.

The modulus of the frequency response parametrized by \( \kappa \) attains its minimum at \( \omega_0 \) for some \( \kappa_0 \), independent of the number of vehicles. This smallest modulus at \( \omega_0 \) is denoted as \( \zeta_{\text{min}} > 1 \) and it is unchanged for any and all diagonal blocks. By Theorem 2, the blocks are connected in series, therefore the total gain of the platoon is given by a product

\[
|T_N(j\omega_0)| = \prod_{i=2}^N |T_{\lambda_i}(j\omega_0)| \geq (\zeta_{\text{min}})^{N-1}.
\]

(17)

The exponential growth of the peak in the magnitude frequency response has thus been proved. Although the eigenvalues of the Laplacian change upon adding more vehicles into the platoon, the bound on eigenvalues as well as the corresponding gain \( \zeta_{\text{min}} \) remain constant.

To summarize, it suffices to test only a single transfer function \( T_{\lambda_{\text{min}}}(s) \) instead of the model of the whole platoon. If this transfer function is larger in \( H_\infty \) norm than one and there is a lower bound on the Fiedler’s eigenvalue, the harmonic instability must occur and cannot be overcome by any linear controller. Note, however, that even systems with only one integrator in the open loop can be harmonically unstable.

IV. Special Cases and Simulations

A particularly important case is when there are two integrators in the open loop.

Lemma 2. For at least two integrators in the open loop, frequency response of each term in the product (12) has a resonance peak, i.e., \( |T_{\lambda_i}(s)| = 1 < |T_{\lambda_i}(j\omega_0)| \).

Proof. Each term in the product in Theorem 2 is a closed-loop transfer function with at least two integrators in the open loop. For such system it was proved in Theorem 1 in [1] that it must have \( H_\infty \) norm greater than 1.

Using the fact that \( ||T_{\lambda_{\text{min}}}(s)||_\infty > 1 \) with at least two integrators in the open loop, we satisfy the conditions in Theorem 3 and can extend the results of [1].

Corollary 2. Vehicular platoon with uniformly bounded eigenvalues of Laplacian and at least two integrators in the open loop is harmonically unstable. This cannot be cancelled by any linear controller.

Theorem 1 proves uniform bound for arbitrary asymmetric formation, so we can extend results of [11] to varying asymmetry and arbitrary dynamical models with two integrators.

Corollary 3. Asymmetric bidirectional control with \( \epsilon_i < \epsilon_{\text{max}} < 1 \forall i, \forall N \) and with at least two integrators in the open loop is harmonically unstable.

It was proved in [11], [18] that if the asymmetric platoon uses identical asymmetries \( \epsilon_i = \epsilon, \mu_i = 1, \) the eigenvalues of Laplacian are given in closed form as \( \lambda_i = -2\sqrt{\epsilon} \cos \theta_i + 1 + \epsilon, \) where \( \theta_i \) is given as the ith solution of the nonlinear equation \( \sin(N\theta_i) - \sqrt{\frac{1}{\epsilon}} \sin((N+1)\theta_i) \) on the interval \((0, \pi)\). The Laplacian eigenvalues are thus bounded and the bounds \( \lambda_{\text{min}} \geq (-1 - \sqrt{\epsilon})^2, \lambda_{\text{max}} \leq (1 + \sqrt{\epsilon})^2 \) do not depend on \( N \). Such formation satisfies the conditions of Corollary 3.

Another special case, which is harmonically unstable, is the predecessor following algorithm with \( \epsilon_i = 0, \forall i \). On the other hand, harmonic stability of symmetric bidirectional control \( (\epsilon_i = 1) \) for double integrator model was proved in [12].

The simulation results comparing the asymmetric control with \( \epsilon_i = 0.5 \) and the symmetric control with \( \epsilon_i = 1 \) are shown in Fig. 3a and 3b. For the asymmetric control scheme it is apparent that as the number of vehicles grows, the peak in the magnitude frequency response grows exponentially and it is localized at almost identical and wide frequency range for any number of cars. The figure 3c shows step response of the platoon, which is oscillating and has very high overshoot. The models used in all cases are \( R(s) = 110s^5 + 43s + 3, G(s) = \frac{1}{s} \). The controller has been chosen so that the overall system is asymptotically stable for any number of vehicles.

V. Conclusion

We dealt with a vehicular platoon controlled in a distributed and asymmetric way where each vehicle only measures the distance to its immediate neighbors. We studied harmonic instability of the platoon, which is a term for exponential scaling of the \( H_\infty \) norm of the transfer function of the platoon as the number of vehicles in the platoon grows.
Let \( \lambda_i \) be transformed by the reduced Laplacian \( L_r \) into a diagonally dominant form \( B = P^{-1} L_r P \). After the transformation, each row of \( B \) reads
\[
[\ldots 0 - \frac{p_{i-1}}{p_i} \mu_i \mu_i(1 + \epsilon_i) - \frac{p_{i+1}}{p_i} \mu_i \epsilon_i 0 \ldots].
\] (19)
To make it diagonally dominant, it must hold
\[- \frac{p_{i-1}}{p_i} \mu_i + \mu_i(1 + \epsilon_i) - \frac{p_{i+1}}{p_i} \mu_i \epsilon_i \geq 0 \quad \forall i.
\] (20)
This is a difference inequality with variable \( p \). We take \( p \) as
\[ p = \frac{1}{2} \left( 1 + \frac{1}{\epsilon_{\max}} \right), \] (21)
which satisfies the inequality. Then \( P \) is a diagonal matrix \( P = \text{diag}(1, p, p^2, \ldots, p^{N-2}) \). Applying this transformation to \( L_r \), we get the \( i \)th row
\[
[\ldots 0 - \frac{1}{p} \mu_i \mu_i(1 + \epsilon_i) - p \mu_i \epsilon_i 0 \ldots].
\] (22)
The sum in each row equals the distance \( d_i = \mu_i(1 + \epsilon_i) - \frac{1}{p} \mu_i - p \mu_i \epsilon_i \) of Geršgorin’s circle from zero and should be positive. After simple calculations, we obtain
\[
d_i = \mu_i \left[ \frac{\epsilon_i - \epsilon_{\max}}{2} + \frac{1 - \epsilon_{\max}}{1 + \epsilon_{\max}} \right].
\] (23)
Assume, without loss of generality, that \( \mu_i \geq 1 \). Then \( d_i \) in the equation above is minimized for \( \epsilon_i = \epsilon_{\max} \). Therefore, the smallest distance of Geršgorin disks from zero, hence also the lower bound on the eigenvalues is
\[
\lambda_{\min} \geq -\frac{1 - \epsilon_{\max}}{2} + \frac{1 - \epsilon_{\max}}{1 + \epsilon_{\max}} = \frac{(1 - \epsilon_{\max})^2}{2 + 2\epsilon_{\max}}.
\] (24)
Furthermore, it is positive for any \( \epsilon_i \leq \epsilon_{\max} \), making \( B \) diagonally dominant. To summarize, we found a bound which does not depend on the matrix size. \( \Box \)

**Appendix B**

**Proof of Theorem 2**

Before the proof, we need the following technical result.

**Lemma 4.** Let \( h_i = g_i v_{N_i} \). Then we have
\[
\sum_{i=2}^{N} h_i^2 = 0 \quad \text{for } m = 0, 1, \ldots, N - 3,
\] (25)
\[
\sum_{i=2}^{N} h_i \frac{1}{\lambda_i} = \frac{1}{\mu_2}.
\] (26)

**Proof.** The terms in \( h_i = g_i v_{N_i} \) can be written as \( g_i = c_i^T V^{-1} e_2 \) and \( v_{N_i} = e_N^T V \). Plugging them into (25) yields
\[
\sum_{i=2}^{N} h_i \lambda_i^m = c_N^T V J M V^{-1} e_2 = c_N^T L^m e_2 = (L^m)_{N2}(27)
\]
where \( (L^m)_{ij} \) is the \((i, j)\) element of \( L^m \). Laplacian is a banded matrix with nonzero diagonal and the first subdiagonal and by powering it, we add new nonzero bands. Hence, \( L \) can be powered at most \( N - 3 \) times to keep zeros at

**Appendix A**

**Proof of Theorem 1**

Before we proceed to the proof, we state one useful Lemma.

**Lemma 3.** [15, Cor. 6.1.6] Let \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \) and let \( p_1, \ldots, p_n \) be positive numbers. Consider the matrix \( B = P^{-1} A P \) with \( P = \text{diag}(p_1, \ldots, p_n) \) and \( b_{ij} = [p_i a_{ij} / p_i] \). Then all eigenvalues of \( A \) lie in the union of Geršgorin disks
\[
\bigcup_{i=1}^{n} \{ z \in \mathbb{C} : |z - a_{ii}| \leq \frac{1}{p_i} \sum_{j=1,j \neq i}^{n} p_j |a_{ij}| \}. \] (18)
\((N-2)\)th subdiagonal and the element \((L^m)_{N2} = 0\) for \(m = 0, \ldots, N - 3\).

Let \(J_r = V_r^{-1}L_rV_r\) be the Jordan form of \(L_r\). Equation (25) is obtained in a similar way using \(V_rJ_r^{-1}V_r^{-1} = L_r^{-1}\) as

\[
\sum_{i=2}^{N} h_i \frac{1}{\lambda_i} = e^{T_{N-1}J_r} e_1 = (L_r^{-1})_{N-1,1} = \frac{1}{\mu_2}.
\]

(28)

Proof of Theorem 2. For simplicity, only the case of non-defective Laplacian is shown here. First we need to evaluate a characteristic polynomial of \(L_r\):

\[
\det(sI_{N-1} + L_r) = s^{N-1} + \alpha_{N-2}s^{N-2} + \ldots + \alpha_1 s + \alpha_0,
\]

(29)

where \(\alpha_{N-2} = \text{Tr}(L_r) = \sum_{i=2}^{N} \lambda_i\) and \(\alpha_0 = \prod_{i=2}^{N} \lambda_i\).

The transfer function \(T_N(s)\) was defined in (11) as

\[
T_N(s) = \sum_{i=1}^{N} g_i F_i(s) v_{N,i} = \sum_{i=1}^{N} g_i v_{N,i} = \frac{b(s)q(s)}{a(s)p(s) + \lambda_i b(s)q(s)}.
\]

(30)

Since \(g_1 = 0\) (the leader cannot be controlled from the second vehicle), the block corresponding to \(\lambda_1 = 0\) does not enter the sum (30), which then has \(N - 1\) terms and reads

\[
T_N(s) = \sum_{i=2}^{N} h_i \psi \prod_{j=2, j \neq i}^{N} [\phi + \lambda_j \psi] = \frac{\phi^{N-2}}{\sum_{i=2}^{N} h_i \psi \prod_{j=2, j \neq i}^{N} [\phi + \lambda_j \psi]}.
\]

(31)

We define \(\phi(s) = a(s)p(s), \psi(s) = b(s)q(s)\) and \(h_i = g_i v_{N,i}\). The argument \(s\) is omitted. The numerator of (31) is then

\[
\sum_{i=2}^{N} h_i \psi \prod_{j=2, j \neq i}^{N} [\phi + \lambda_j \psi] = \sum_{i=2}^{N} h_i \psi \left\{ \phi^{N-2} + \phi^{N-3} \sum_{j=2, j \neq i}^{N} \lambda_j + \phi^{N-4} \sum_{j=2, j \neq i}^{N} \lambda_j \lambda_k + \ldots \right\}.
\]

(32)

Let us put the terms with equal powers of \(\phi^j \psi^k\) in (32) together. First, take those with \(\phi^{N-2} \psi^k\). The sum \(\phi^{N-2} \psi^k \sum_{i=2}^{N} h_i = 0\), using (25) in Lemma 4. Second, take those with \(\phi^{N-3} \psi^k\):

\[
\phi^{N-3} \psi^k \sum_{i=2}^{N} h_i \prod_{j=2, j \neq i}^{N} \lambda_j = \phi^{N-3} \psi^k \sum_{i=2}^{N} h_i (\alpha_{N-2} - \lambda_i) = \phi^{N-3} \psi^k \prod_{i=2}^{N} \lambda_i = 0.
\]

(33)

We used the fact that \(h_1 (\lambda_2 + \ldots + \lambda_{i-1} + \lambda_{i+1} + \ldots + \lambda_N) = h_1 (\alpha_{N-2} - \lambda_i)\) and then applied Lemma 4. Using similar constructions we arrive to the fact that all powers of \(\phi^j \psi^k\) are multiplied by zero, so they vanish. The only exception is the term with \(\psi^{N-1}\), for which we get

\[
\psi^{N-1} \sum_{i=2}^{N} h_i \prod_{j=2, j \neq i}^{N} \lambda_i = \psi^{N-1} \alpha_0 \sum_{i=2}^{N} h_i \frac{1}{\lambda_i} = \psi^{N-1} \frac{\alpha_0}{\mu_2}.
\]

(34)

We used the fact that \(\prod_{i=2}^{N} \lambda_i = \alpha_0\), so \(\prod_{j=2, j \neq i}^{N} \lambda_j = \frac{\alpha_0}{\lambda_i}\). The last equality follows from (26). With all these terms, the fraction in (31) simplifies to

\[
T_N(s) = \frac{1}{\mu_2} \frac{\prod_{i=2}^{N} (b(s)q(s)) \prod_{i=2}^{N} \lambda_i}{\sum_{i=2}^{N} a(s)p(s) + \lambda_i b(s)q(s)}.
\]

(35)

This is a series interconnection of blocks \(T_{\lambda_i}(s)\), which proves the theorem. The cases with defective Laplacian matrices can be treated in a similar way with all the results valid. □

References