Scaling of H-infinity Norm in Symmetric Bidirectional Platoons [⋆]

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Abstract: In this paper scaling of H-infinity norm of selected transfer functions in bidirectional vehicle platoons is investigated. The vehicles are identical and use symmetric nearest-neighbor interactions, hence the communication topology is a pinned undirected path graph. The LTI open-loop model of individual vehicles can be of arbitrary order, but it is required to contain two integrators in order to track the movement of the leader. It is shown that for any agent model some transfer functions in the formation scale linearly, while others scale quadratically. The type of scaling depends on the steady-state gain of the transfer function.

Keywords: Distributed control, scaling, LTI models, vehicle platoons, bidirectional control.

1. INTRODUCTION

Symmetric control of bidirectional vehicle platoons recently received a lot of attention. The popularity of symmetric control is partially thanks to its possible applications in real vehicle platoons, but also because it reveals a few fundamental properties of the distributed control.

Asymptotic stability was discussed by Barooah and Hespanha (2005), who proved that such platoons cannot have more than two integrators in the open loop. The same paper also shows that the platoon's response to noise is unbounded in the number of vehicles N. This means a kind of string instability. Effect of noise was also investigated by Bamieh et al. (2012), which shows that one-dimensional graph has the worst scaling of the $\mathcal{H}_2$ norm.

Veerman et al. (2007) showed that the $\mathcal{H}_\infty$ norm of the transfer function from the leader's position to the position of the last vehicle grows linearly with N. The same result was later obtained by Hao and Barooah (2013). Transient properties of a response of the platoon to the step in leader's velocity were derived by Veerman et al. (2009). Knorn et al. (2014) analyzed scaling of bidirectional platoons using port-Hamiltonian framework and Knorn et al. (2015) discussed the effects of measurement errors.

In contrast, to shorten the transient in platoon, Barooah et al. (2009) proposed asymmetric bidirectional control. For one integrator in the open loop the properties can be good (Lin et al. (2012)). However, as shown by Tangerman et al. (2012), for double-integrator models an exponential scaling of $\mathcal{H}_\infty$ norm occurs. Hence, there is a price to pay for faster transients.

All of the aforementioned papers considered only a particular vehicle model. Most typically it was a double integrator or a vehicle controlled by a PI controller. Nevertheless, control theory allows us to design a controller of an arbitrary order and some design approaches yield higher-order controllers. Therefore, it is important to generalize the results from these papers to some general classes of systems. First step in this generalization was by Seiler et al. (2004), in which string stability was disproved for predecessor following when agents have two integrators in the open loop. Similarly, Herman et al. (2015) proved that any asymmetric control has exponential scaling of $\mathcal{H}_\infty$ norm for vehicles with two integrators in the open loop.

In this paper we continue in this line of research. In particular, we generalize the results about scaling of $\mathcal{H}_\infty$ norm of transfer functions in symmetric bidirectional platoons in which vehicles have any LTI model with two integrators in the open loop. Namely, we investigate a norm of two transfer functions: from the input of the first vehicle to position of the last vehicle and from the input of the last vehicle to its own output. We show that the first one scales linearly (as is known in the literature for particular models), while the latter quadratically.

Recent results suggest that partial asymmetry—symmetric coupling in position and asymmetric in velocity (or other states)—achieves short transients with a good scaling (see Herman et al. (2016)). In view of these results, a completely symmetric bidirectional control does not seem as a good solution for a vehicle platoon. On the other hand, the theoretical analysis of the systems with partial asymmetry is still very limited and even a stability analysis is a difficult task. The reason for this difficulty is that the Laplacians for position and velocity are not simultaneously diagonalizable. The authors therefore believe that even in platoons it is useful to fully understand the implications of symmetry for scaling. Moreover, the results for partial asymmetry were derived only for particular models, while here we provide results for general models.

2. SYSTEM MODEL

Assume a string of $N + 1$ *identical* vehicles indexed $i = 0, 1, \ldots, N$, having their outputs (positions) in one-

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dimensional space. Except for platoons, one might consider also a mass-spring-damper chain as an example. The index 0 corresponds to the leader, which is controlled independently of the rest of the platoon. In this paper we work with an LTI open-loop model of the agent. That is, a dynamic controller $R(s) = \frac{q(s)}{p(s)}$ is connected in series with a vehicle (plant) model $G(s) = \frac{b(s)}{a(s)}$. The output of the ith vehicle is denoted as y_i and it is the position of the vehicle. The open-loop model is then $M(s) = R(s)G(s) = \frac{b(s)q(s)}{a(s)p(s)}$. We only require that the open loop is a proper transfer function, otherwise its order and structure are not limited.

Factor the open loop as $M(s) = \frac{1}{s^\eta} M_\mathrm{s}(s)$ with $|M_\mathrm{s}(0)| < \infty$. Then η is the number of integrators in the open loop (also called a type number of the system). For instance, the model $M(s) = \frac{s+1}{s^2(s+b)}$ has $\eta = 2$. For $\eta = 2$ and $M_\mathrm{s}(s) = 1$, we get the well-known double-integrator system.

Each vehicle uses for its control the outputs of its two nearest neighbors—its predecessor y_{i-1} and its successor y_{i+1}— and an external reference r_i. The control law is

$$u_i = (y_{i-1} - y_i) - (y_i - y_{i+1}) + r_i \tag{1}$$

and the last agent uses the control law $u_N = (y_{N-1} - y_N) + r_N$. The general external input r_i can represent, for instance, a disturbance acting at the input of the controller or a reference such as the reference distance Δ_ref. The leader's control input is just r_0.

2.1 Properties of the Laplacian

The regulation errors in (1) are given in a vector form as $u = -Ly + r$ with $u = [u_0, \ldots, u_N]^\mathrm{T}$, $y = [y_0, \ldots, y_N]^\mathrm{T}$ and $r = [r_0, \ldots, r_N]^\mathrm{T}$. The matrix $L = [l_{ij}] \in \mathbb{R}^{N+1 \times N+1}$ is the Laplacian of a path graph with a leader. L has the following structure

$$L = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ -1 & 2 & -1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & -1 & 2 & -1 \\ 0 & \ldots & 0 & -1 & 1 \end{bmatrix}. \tag{2}$$

Except for the first line, it is a symmetric tridiagonal matrix. Next we state some useful properties of L.

Lemma 1. Laplacian L in (2) and its eigenvalues λ_i, $i = 0, 1, \ldots, N$ have the following properties:

a) Let L_p be the matrix obtained from L by deleting the first row and column. Then $\lambda_i(L) = \lambda_i(L_\mathrm{p}), \forall i \geq 1$.
b) The eigenvalues of L can be calculated as $\lambda_0 = 0$ and
$$\lambda_i = 2\left(1 - \cos\left(\frac{(2i-1)\pi}{2N+1}\right)\right) = 4\sin^2\left(\frac{(2i-1)\pi}{4N+2}\right). \tag{3}$$
c) Let $\bar{L}_{kk}$ be a matrix obtained from L_p by deleting the kth row and column. Let the eigenvalues of $\bar{L}_{kk}$, $1 \leq k \leq N$, be $\gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_{N-1}$. Then $\lambda_i \leq \gamma_i \leq \lambda_{i+1}$.
d) The kth eigenvalue λ_k is bounded as
$$\frac{4(2k-1)^2}{(2N+1)^2} \leq \lambda_k \leq \frac{(2k-1)^2\pi^2}{4N^2} \tag{4}$$
and for $k \ll N$
$$\lambda_k \approx \frac{(2k-1)^2\pi^2}{(4N+2)^2}. \tag{5}$$
Thus, the kth smallest eigenvalue approaches zero as N grows with a quadratic rate.

Proof. a) The pinned Laplacian L_p has a form

$$L_\mathrm{p} = \begin{bmatrix} 2 & -1 & 0 & \ldots \\ -1 & \ddots & \ddots & 0 \\ \ldots & \ddots & 2 & -1 \\ 0 & \ldots & -1 & 1 \end{bmatrix}. \tag{6}$$

The result follows from the fact that L is a block lower-triangular matrix with blocks $[0, L_\mathrm{p}]$. b) The eigenvalues of L_p were calculated in (Parlangeli and Notarstefano, 2012, Prop. 3). c) This is so called Cauchy Interlacing Theorem (Horn and Johnson (1990)). d) From (3) using $\sin x \leq x$ for $x \in (0, \pi/2)$ we get $\lambda_k \leq 4\left(\frac{(2k-1)\pi}{4N+2}\right)^2 \leq \frac{(2k-1)^2\pi^2}{4N^2} = \frac{c_2(k)}{N^2}$. The lower bound follows from (3) using $\sin x \geq 2x/\pi$ as $\lambda_k \geq 4\left(\frac{2}{\pi}\frac{(2k-1)\pi}{4N+2}\right)^2 = \frac{4(2k-1)^2}{(2N+1)^2} \geq \frac{c_1(k)}{N^2}$. The approximation is a consequence of small angle argument $\sin x \approx x$ for small x. □

A similar result on the quadratic approach of the smallest eigenvalue to zero was presented by Barooah et al. (2009).

2.2 Transfer functions

In this paper we investigate the effect of an external input to the position of some (other) vehicle. Consider an input r_c acting at a vehicle with index c (called control vehicle). We are interested in how this input affects position y_o of the vehicle with index o (output vehicle). The transfer function between such input and output is

$$T_{co}(s) = \frac{y_o(s)}{r_c(s)}. \tag{7}$$

We are interested in how the $\mathcal{H}_\infty$ norm of this transfer function changes as a function of the indices c and o. The $\mathcal{H}_\infty$ norm of a stable transfer function $T(s)$ is defined as $\|T(s)\|_\infty = \sup_{\omega\in\mathbb{R}} |T(\jmath\omega)|$. In the sequel we work with the *pinned Laplacian* L_p in (6) in order to get rid of $\lambda_0 = 0$.

It was derived by Herman et al. (2014) that the transfer function $T_{co}(s)$ has the following form

$$T_{co}(s) = \frac{[b(s)q(s)]^{1+\delta_{co}} \prod_{i=1}^{N-1-\delta_{co}} \left(a(s)p(s) + \gamma_i b(s)q(s)\right)}{\prod_{i=1}^{N} \Big(a(s)p(s) + \lambda_i b(s)q(s)\Big)}, \tag{8}$$

where δ_{co} is the graph distance from c to o and λ_i are from (3). The terms $\gamma_i \in \mathbb{R}$, $\gamma_i \leq \gamma_{i+1}$, are the eigenvalues of the matrix $\bar{L}_{co} \in \mathbb{R}^{N-\delta_{co}-1 \times N-\delta_{co}-1}$ that is obtained from L_p by deleting all the rows and columns corresponding to the nodes on the path from c to o, see (Herman et al., 2014, Thm. 10). Note that $\bar{L}_{co}$ is a principal submatrix of L_p, hence γ_i and λ_i interlace in the sense of Lemma 1 c).

We can rewrite the transfer function (8) into a more convenient form. We define two types of transfer functions

$$T_j(s) = \frac{\lambda_j b(s)q(s)}{a(s)p(s) + \lambda_j b(s)q(s)}, \tag{9}$$

$$Z_{ij}(s) = \frac{a(s)p(s) + \gamma_i b(s)q(s)}{a(s)p(s) + \lambda_j b(s)q(s)}. \tag{10}$$

From the product in (8), we can form $\delta_{co}+1$ transfer functions of type $T_j(s)$ and $N-\delta_{co}-1$ of type $Z_{ij}(s)$. So the transfer function can be written as

$$T_{co}(s)=T_{co}(0)\prod_{i=1,j\in\mathcal{J}}^{N-\delta_{co}-1}\frac{\lambda_j}{\gamma_i}Z_{ij}(s)\prod_{j=1,j\notin\mathcal{J}}^{N}T_j(s), \quad (11)$$

where $\mathcal{J}$ is the set of indices j of eigenvalues λ_j which are used in $Z_{ij}(s)$. The term $T_{co}(0)=\frac{\prod_{i=1}^{N-\delta_{co}-1}\gamma_i}{\prod_{j=1}^{N}\lambda_j}$ is the steady-state gain.

Assumption 2. The overall system with the Laplacian L_{p} (without the leader) is asymptotically stable for all N.

This is quite easy to achieve, since we just require that the polynomial $a(s)p(s)+\lambda_j b(s)q(s)$ is stable for any $\lambda_j\in(0,4]$, i.e., for the real interval of eigenvalues. Since the gain λ_i can be arbitrarily close to zero, we cannot stabilize systems which have unstable open loops.

Assumption 3. $M(s)$ has neither poles nor zeros in the closed right half-plane, except for $\eta=2$ poles at the origin.

3. SCALING OF $\mathcal{H}_\infty$ NORM

In this paper we discuss scaling of platoons in which vehicles have open-loop models with $\eta=2$. Two integrators in the open loop are required to satisfy the Internal Model Principle in distributed control, see Lunze (2012).

We will approximate the frequency response of the open loop for low frequencies ω as follows. Let the open loop with $\eta=2$ be written as

$$M(s)=\frac{d_m s^m+d_{m-1}s^{m-1}+\ldots+d_1 s+d_0}{s^2(c_{n-2}s^{n-2}c_{n-3}s^{n-3}+\ldots+c_1 s+c_0)}. \quad (12)$$

Calculating its frequency response, separating to real and imaginary part, we get

$$M(\jmath\omega)=-\frac{d_0c_0+\mathcal{O}(\omega^2)}{\omega^2\left(c_0^2+\mathcal{O}(\omega^2)\right)}-\jmath\frac{d_0c_1-d_1c_0+\mathcal{O}(\omega)}{\omega\left(c_0^2+\mathcal{O}(\omega^2)\right)}. \quad (13)$$

Assuming that ω is small, we can neglect the higher-order terms in the numerator and denominator. Hence the real part $\Re\{M(\jmath\omega)\}\approx-\frac{d_0c_0}{\omega^2c_0^2}$ and the imaginary part $\Im\{M(\jmath\omega)\}\approx\frac{d_0c_1-d_1c_0}{\omega c_0^2}$. Using $g_{\mathrm{r}}=\frac{d_0}{c_0}$ and $g_{\mathrm{i}}=\frac{d_0c_1-d_1c_0}{c_0^2}$, the approximated open-loop frequency response reads

$$\overline{M}(\jmath\omega)=-g_{\mathrm{r}}\frac{1}{\omega^2}-\jmath g_{\mathrm{i}}\frac{1}{\omega}. \quad (14)$$

We also need the following approximation of a square root $\sqrt{1+x}\approx1+\frac{1}{2}x-\frac{1}{8}x^2+\ldots$. When x is very small, then

$$\sqrt{1+x}\approx1+\frac{1}{2}x. \quad (15)$$

In order to show scaling of the $\mathcal{H}_\infty$ norm of transfer functions in formation, we begin with an auxiliary result. Consider the transfer function

$$T_1(s)=\frac{\lambda_1 b(s)q(s)}{a(s)p(s)+\lambda_1 b(s)q(s)}. \quad (16)$$

This is a closed-loop system with a gain equal to λ_1. This gain gets to zero as λ_1 approaches zero.

Lemma 4. Suppose that $\eta=2$. Then for N large and for any $M(s)$ it holds for some $c_1\le c_2\in\mathbb{R}$,

$$c_1N\le\|T_1(s)\|_\infty\le c_2N. \quad (17)$$

The proof is in Appendix A. This means that the peak in the magnitude frequency response of a closed loop corresponding to the smallest nonzero eigenvalue of the Laplacian scales linearly with the number of vehicles.

3.1 Particular transfer functions

In this section we analyze norms of two important transfer functions, which capture the effects of the inputs on both ends of the platoon. First, it is the transfer function $T_{1,N}(s)=\frac{y_N(s)}{r_1(s)}$. This transfer function is quite important, as it describes the effect of the movement of the leader on the last vehicle. The second one under consideration is the transfer function $T_{N,N}(s)$ from the input of the last agent to its own position. The reason why we selected this transfer function is that it has the worst scaling in the platoon. Moreover, it also captures the effect of the rear-end input on the platoon.

First we analyse the steady-state gain of transfer functions.

Corollary 5. The steady-state gain $|T_{co}(0)|$ is given as

$$|T_{co}(0)|=\begin{cases}c & \text{if } c\le o,\\ o & \text{if } c<o.\end{cases} \quad (18)$$

Proof. This is a simple consequence of (Herman et al., 2017, Thm. 1). □

That is, the steady-state gain is just the index of the control or output vehicle. Hence, if $c=N$, then the steady state gain is equal to N and it also grows as N grows.

The norm of the transfer function $T_{1,N}(s)$ scales as follows.

Theorem 6. Suppose that $\eta=2$. Then for N large, the norm of the transfer function $\|T_{1,N}(s)\|_\infty$ for any $M(s)$ scales linearly in N, i.e., for some $c_1\le c_2\in\mathbb{R}$,

$$c_1N\le\|T_{1,N}(s)\|_\infty\le c_2N. \quad (19)$$

The proof is in Appendix B. For the double-integrator model, linear scaling was proved by Hao and Barooah (2013) and Veerman et al. (2007). Theorem 6 therefore generalizes the result to an arbitrary open-loop model.

Next we show the scaling of $\|T_{N,N}(s)\|_\infty$.

Theorem 7. Suppose that $\eta=2$. Then for N large, the norm of the transfer function $\|T_{N,N}(s)\|_\infty$ for any $M(s)$ scales quadratically in N, i.e., for some $c_1\le c_2\in\mathbb{R}$,

$$c_1N^2\le\|T_{N,N}(s)\|_\infty\le c_2N^2. \quad (20)$$

The proof is in Appendix C. This is basically a consequence of the linear scaling of the peak of $T_1(s)$ and the linear scaling of the steady-state gain $T_{N,N}(0)$ as N grows. This transfer function has the greatest norm among all in the platoon.

Remark 8. Due to space limitations, we do show the results for $\eta=1$. We just state here that $\|T_{1,N}(s)\|_\infty$ is bounded and that $\|T_{N,N}(s)\|_\infty$ scales linearly. Another practically relevant norm is the norm of the transfer function matrix $\mathbf{T}(s)$, for which it holds $\|\mathbf{T}(s)\|_\infty\propto N^2$ for $\eta=1$ and $\|\mathbf{T}(s)\|_\infty\propto N^3$ for $\eta=2$.

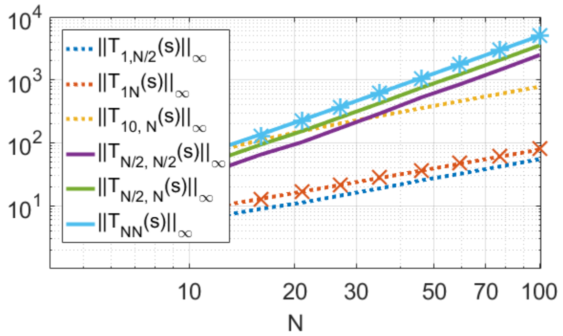

Fig. 1. The norms of several transfer functions in logarithmic scale with N growing. The crosses are $0.8N$ and the asterisks are $0.6N^2$. The norm was calculated using *norm* function in Matlab.

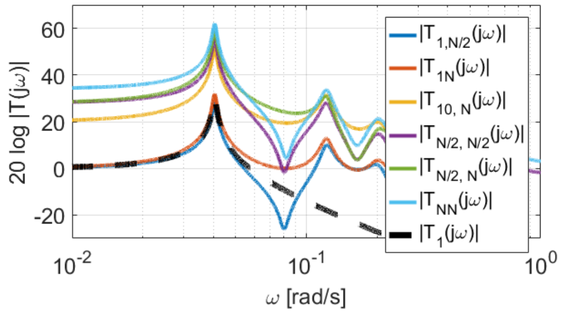

Fig. 2. Frequency response of several transfer functions.

4. NUMERICAL EXAMPLES

We verify the results using the following models:

$$R(s) = \frac{6.9s^3 + 33s^2 + 49s + 23}{s^3 + 4.5s^2 + 4.5s}, \quad G(s) = \frac{1}{s(s+3)}. \quad (21)$$

This is a higher-order controller with an integral action designed for a second-order vehicle model with friction.

The scaling of various transfer functions in platoon is illustrated in Fig. 1. It is apparent from the figure that whenever the index of the control node c is kept fixed, the scaling is linear. This is caused by the bounded steady-state gain (for fixed c the steady-state gain (18) is fixed) and by the linear growth of the peak $T_1(s)$. These are the transfer functions $T_{1,N/2}(s), T_{1,N}(s), T_{10,N}(s)$. However, when the index c grows with the number of agents, the linear growth of $\|T_1(s)\|_\infty$ combines with the linear growth of $|T_{co}(0)|$, therefore the scaling becomes quadratic, as it is apparent for transfer functions $T_{N/2,N/2}(s), T_{N/2,N}(s)$ and $T_{N,N}(s)$. The crosses confirm that $\|T_{1,N}(s)\|_\infty$ scales linearly and asterisks that $\|T_{N,N}(s)\|_\infty$ scales quadratically.

Magnitude frequency responses of the same transfer functions for $N = 50$ are shown in Fig. 2. It can be seen that all the transfer functions have their maximum at the frequency where $T_1(\jmath\omega)$ has its peak. Also the shape and height of the peak for all transfer functions is similar to the height and shape of $T_1(\jmath\omega)$. This is a direct consequence of the product form (11), where the term $T_1(s)$ is present for any combination of c and o. The value of the peak differs mainly due to different steady-state gains.

5. CONCLUSION

This paper considered scaling of $\mathcal{H}_\infty$ norm of transfer functions in symmetric bidirectional platoons. The vehicles are supposed to be modelled by identical LTI systems having two integrators in the open loop. It was proved that the norm of the transfer function from the first vehicle to the last vehicle grows linearly with the number of vehicles, while the norm of the transfer function from the input of the last vehicle to its own position grows quadratically. The results presented here generalize the results previously obtained for double-integrator models to arbitrary linear model of the vehicle.

Appendix A. PROOF OF LEMMA 4

Proof. As $\lambda_1 \to 0$, the bandwidth of the closed loop $T_1(s) = \frac{\lambda_1 b(s)q(s)}{a(s)p(s)+\lambda_1 b(s)q(s)}$ decreases, because λ_1 acts as a gain. Thus, the maximum of $|T_1(\jmath\omega)|$ can be only in low frequencies. The approximated frequency response of $\overline{T}_1(\jmath\omega) = \frac{\lambda_1 \overline{M}(\jmath\omega)}{1+\lambda_1 \overline{M}(\jmath\omega)}$ is using $\overline{M}(s)$ in (14)

$$\left|\overline{T}_1(\jmath\omega)\right|^2 = \frac{\lambda_1\left(g_\mathrm{i}^2\omega^2 + g_\mathrm{r}^2\right)}{\left(g_\mathrm{r}\lambda_1 - \omega^2\right)^2 + \lambda_1^2 g_\mathrm{i}^2\omega^2}. \quad (A.1)$$

We can calculate the frequency ω_m for which (A.1) attains its maximum. It is

$$\omega_\mathrm{m} = \frac{g_\mathrm{r}}{g_\mathrm{i}}\sqrt{\sqrt{2\lambda_1 g_\mathrm{i}^2/g_\mathrm{r} + 1} - 1}. \quad (A.2)$$

Plugging this frequency for ω to (A.1), we get

$$\max_\omega |\overline{T}(\jmath\omega)|^2 = \frac{g_\mathrm{i}^4\sqrt{g_\mathrm{r} + 2g_\mathrm{i}^2\lambda_1}\lambda_1^2}{\psi(\lambda_1)}, \quad (A.3)$$

where $\psi(\lambda_1) = g_\mathrm{i}^4\sqrt{g_\mathrm{r}+2g_\mathrm{i}^2\lambda_1}\lambda_1^2 + \Big(-2g_\mathrm{r}g_\mathrm{i}^2\sqrt{g_\mathrm{r}+2g_\mathrm{i}^2\lambda_1} + 4g_\mathrm{i}^2\sqrt{g_\mathrm{r}^3}\Big)\lambda_1 - 2g_\mathrm{r}^2\sqrt{g_\mathrm{r}+2g_\mathrm{i}^2\lambda_1} + 2\sqrt{g_\mathrm{r}^5}$. Define $\tau = \sqrt{2g_\mathrm{i}^2\lambda_1 + g_\mathrm{r}}$ and $\tau^2 = 2g_\mathrm{i}^2\lambda_1 + g_\mathrm{r}$. Then we simplify (A.3) to

$$\max_\omega |\overline{T}(\jmath\omega)|^2 = \frac{g_\mathrm{i}^4\lambda_1^2\tau}{g_\mathrm{i}^4\lambda_1^2\tau + 2g_\mathrm{r}\sqrt{g_\mathrm{r}}(2g_\mathrm{i}^2\lambda_1 + g_\mathrm{r}) - g_\mathrm{r}\tau(2g_\mathrm{i}^2\lambda_1 + g_\mathrm{r}) - g_\mathrm{r}^2\tau}$$
$$= \frac{g_\mathrm{i}^4\lambda_1^2\tau}{g_\mathrm{i}^4\lambda_1^2\tau + 2g_\mathrm{r}\sqrt{g_\mathrm{r}}\tau^2 - g_\mathrm{r}\tau^3 - g_\mathrm{r}^2\tau} = \frac{g_\mathrm{i}^4\lambda_1^2}{g_\mathrm{i}^4\lambda_1^2 - \left(\sqrt{g_\mathrm{r}}\tau - g_\mathrm{r}\right)^2}$$
$$= \frac{g_\mathrm{i}^4\lambda_1^2}{\left(g_\mathrm{i}^2\lambda_1 + g_\mathrm{r}\sqrt{1+\frac{2g_\mathrm{i}^2}{g_\mathrm{r}}\lambda_1} - g_\mathrm{r}\right)\left(g_\mathrm{i}^2\lambda_1 - g_\mathrm{r}\sqrt{1+\frac{2g_\mathrm{i}^2}{g_\mathrm{r}}\lambda_1} + g_\mathrm{r}\right)}. \quad (A.4)$$

We can use (15) to get $\sqrt{1 + \frac{2g_\mathrm{i}^2}{g_\mathrm{r}}\lambda_1} \approx 1 + \frac{g_\mathrm{i}^2}{g_\mathrm{r}}\lambda_1 - \frac{g_\mathrm{i}^4}{2g_\mathrm{r}^2}\lambda_1^2$. Plugging this to (A.4) we get $\max_\omega |\overline{T}_1(\jmath\omega)|^2 = \frac{2g_\mathrm{r}}{2g_\mathrm{i}^2\lambda_1 - \frac{g_\mathrm{i}^4}{2g_\mathrm{r}}\lambda_1^2}$, from which after neglecting the term with λ_1^2 follows the final result

$$\max_\omega |\overline{T}_1(\jmath\omega)| \approx \sqrt{\frac{g_\mathrm{r}}{g_\mathrm{i}^2\lambda_1}} = \sqrt{\frac{g_\mathrm{r}}{g_\mathrm{i}^2}}\frac{c}{N}, \quad (A.5)$$

because the eigenvalue λ_1 is in the order of $1/N^2$. Numerical verification is in Fig. A.1. □

Appendix B. PROOF OF THEOREM 6

Proof. Using (18), equation (11) gets a form

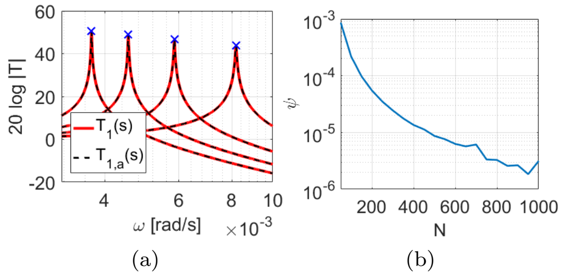

Fig. A.1. a) Frequency responses of $|T_1(\jmath\omega)|$ (red) and $|\overline{T}_1(\jmath\omega)|$ (black) for $N = 250, 350, 450, 550$ (right to left). The blue crosses are from (A.5). b) shows relative error $\psi = \left|\frac{\|T_1(s)\|_\infty - \|\overline{T}_1(s)\|_\infty}{\|T_1(s)\|_\infty}\right|$ of the approximation as a function of N. It decays to zero.

$$T_{1,N}(s) = \prod_{i=1}^{N} T_i(s) = T_1(s)\prod_{i=2}^{N} T_i(s) \tag{B.1}$$

Let ω_m be the frequency at which $T_1(s)$ attains its maximum. It follows from (Herman et al., 2017, Lem. 2 a)) that at this frequency for all transfer functions $T_i(s)$, $i > 1$ holds $|T_i(\jmath\omega_\mathrm{m})| \geq 1$ since $\lambda_i \geq \lambda_1$. Then $\prod_{i=2}^{N} T_i(\jmath\omega_\mathrm{m}) \geq 1$. Also Lemma 4 shows that $\|T_1(s)\|_\infty$ grows linearly in N. Then from (B.1) we get a lower bound

$$\|T_{1,N}(s)\|_\infty \geq \prod_{i=1}^{N} |T_i(\jmath\omega_\mathrm{m})| \geq |T_1(\jmath\omega_\mathrm{m})| \geq c_1 N. \tag{B.2}$$

Now we show that the norm is upper bounded by $c_2 N$ by showing that $\prod_{i=2}^{N} |\overline{T}_i(\jmath\omega_\mathrm{m})|$ is upper bounded with a bound independent of N. Define $\overline{T}_i(\jmath\omega) = \frac{\lambda_i \overline{M}(\jmath\omega)}{1+\lambda_i \overline{M}(\jmath\omega)}$. The frequency response $|\overline{T}_i(\jmath\omega_\mathrm{m})|^2$ at ω_m in (A.2) is given as

$$\left|\overline{T}_i(\jmath\omega_\mathrm{m})\right|^2 = \frac{\lambda_i^2 g_\mathrm{i}^4 \tau}{(\tau - \sqrt{g_\mathrm{r}})g_\mathrm{i}^4\lambda_i^2 + \sqrt{g_\mathrm{r}}(g_\mathrm{i}^2\lambda_i - \sqrt{g_\mathrm{r}}\tau + g_\mathrm{r})^2} \tag{B.3}$$

with $\tau = \sqrt{2g_\mathrm{i}^2\lambda_1 + g_\mathrm{r}}$. Since λ_1 is small, using (15) we approximate $\tau \approx \sqrt{g_\mathrm{r}}\left(1 + \frac{g_\mathrm{i}^2}{g_\mathrm{r}}\lambda_1\right)$. Then

$$\left|\overline{T}_i(\jmath\omega_\mathrm{m})\right|^2 \approx \frac{g_\mathrm{i}^6\lambda_i^2\lambda_1 + g_\mathrm{i}^4 g_\mathrm{r}\lambda_i^2}{g_\mathrm{i}^6\lambda_i^2\lambda_1 + g_\mathrm{i}^4 g_\mathrm{r}(\lambda_i - \lambda_1)^2} = \frac{\lambda_1 + \frac{g_\mathrm{r}}{g_\mathrm{i}^2}}{\lambda_1 + \frac{g_\mathrm{r}}{g_\mathrm{i}^2}(1 - \frac{\lambda_1}{\lambda_i})^2}$$
$$\approx \frac{\frac{g_\mathrm{r}}{g_\mathrm{i}^2}}{\frac{g_\mathrm{r}}{g_\mathrm{i}^2}(1 - \frac{\lambda_1}{\lambda_i})^2} = \frac{1}{(1 - \frac{\lambda_1}{\lambda_i})^2} \Rightarrow \left|\overline{T}_i(\jmath\omega_\mathrm{m})\right| \approx \frac{1}{1 - \frac{\lambda_1}{\lambda_i}}. \tag{B.4}$$

We can bound λ_1/λ_i using (4) as

$$\frac{\lambda_1}{\lambda_i} \leq \frac{\frac{\pi^2}{(4N+2)^2}}{\frac{(2i-1)^2}{(2N+1)^2}} = \frac{\pi^2}{(4i-2)^2}. \tag{B.5}$$

Then $\left|\overline{T}_i(\jmath\omega_\mathrm{m})\right| \leq 1 + \frac{\pi^2}{(4i-2)^2 - \pi^2}$. The product then is

$$\prod_{i=2}^{N}|\overline{T}_i(\jmath\omega_\mathrm{m})| \approx \prod_{i=2}^{N}\frac{1}{1 - \frac{\lambda_1}{\lambda_i}} \leq \prod_{i=2}^{N}\left(1 + \frac{\pi^2}{(4i-2)^2 - \pi^2}\right)$$
$$\leq \prod_{i=2}^{N}\left(1 + \frac{2}{i^2}\right) \leq \prod_{i=1}^{\infty}\left(1 + \frac{2}{i^2}\right) \tag{B.6}$$

The product can be bounded as (Melnikov (2011))

$$\prod_{i=1}^{\infty}\left(1 + \frac{2}{i^2}\right) = \frac{\sinh\sqrt{2}\pi}{\sqrt{2}\pi} = c_2. \tag{B.7}$$

It follows that $|T_{1,N}(\jmath\omega_\mathrm{m})|$ is bounded as

$$|T_{1,N}(\jmath\omega_\mathrm{m})| \leq |T_1(\jmath\omega_\mathrm{m})|\prod_{i=2}^{N}|\overline{T}_i(\jmath\omega_\mathrm{m})| \leq |T_1(\jmath\omega_\mathrm{m})|c_2 \leq c_2 N \tag{B.8}$$

For other peaks caused by blocks $T_i(s)$ in (B.1) we know that their value is lower than that of $T_1(s)$. Moreover, at higher frequencies the blocks $T_i(s)$ with λ_i very low have they roll-off, hence the frequency response is sufficiently low. So it is the peak of $T_1(s)$ that sets the $\mathcal{H}_\infty$ norm and this peak scales linearly by (B.2) and (B.8). □

Appendix C. PROOF OF THEOREM 7

Proof. Consider the real part of the frequency response $\Re\{\lambda_1 M(\jmath\omega_m)\}$ with ω_m in (A.2). Using (14) and (15) it is

$$\alpha = \Re\{\lambda_1 M(\jmath\omega_\mathrm{m})\} \approx \frac{-g_\mathrm{i}^2\lambda_1}{g_\mathrm{r}\left(\sqrt{1 + \lambda_1\frac{2g_\mathrm{i}^2}{g_\mathrm{r}}} - 1\right)}$$
$$\approx \frac{-g_\mathrm{i}^2\lambda_1}{g_\mathrm{i}^2\lambda_1 - \frac{g_\mathrm{i}^4}{g_\mathrm{r}^2}\lambda_1^2} \leq -1 \text{ for } \lambda_1 \to 0, \tag{C.1}$$

since $g_\mathrm{i}, g_\mathrm{r}$ are fixed. This is the frequency response of the open loop scaled by the smallest gain possible, λ_1. Hence, $\lambda_i\overline{M}(\jmath\omega_\mathrm{m}) \leq -1, \forall i$. Using (11), we get $T_{NN}(s)$ as

$$T_{NN}(s) = T_{N,N}(0)T_1(s)\frac{\lambda_N}{\gamma_1}Z_{1N}(s)\prod_{i=2}^{N-1}\left(\frac{\lambda_i}{\gamma_i}Z_{ii}(s)\right), \tag{C.2}$$

where $Z_{1N} = \frac{a(s)p(s)+\gamma_1 b(s)q(s)}{a(s)p(s)+\lambda_N b(s)q(s)}$. It follows from (Herman et al., 2017, Lem. 2 c)) that $\prod_{i=2}^{N-1}\frac{\lambda_i}{\gamma_i}Z_{ii}(\jmath\omega_\mathrm{m}) \geq \prod_{i=2}^{N-1}\frac{\lambda_i}{\gamma_i}Z_{ii}(0) = 1$, because $\lambda_i\alpha < -1$. Therefore, we get using (A.5) and (18)

$$T_{NN}(\jmath\omega_\mathrm{m}) \geq T_{NN}(0)T_1(\jmath\omega_\mathrm{m})\frac{\lambda_N}{\gamma_1}Z_{1N}(\jmath\omega_\mathrm{m})$$
$$\geq c_1 N^2\frac{\lambda_N}{\gamma_1}Z_{1N}(\jmath\omega_\mathrm{m}). \tag{C.3}$$

Let us now test if $\frac{\lambda_N}{\gamma_1}Z_{1N}(\jmath\omega_\mathrm{m})$ is bounded. Define $\overline{Z}_{1N}(\jmath\omega_\mathrm{m}) = \frac{1+\gamma_1\overline{M}(\jmath\omega_\mathrm{m})}{1+\lambda_N\overline{M}(\jmath\omega_\mathrm{m})}$ as an approximation of $Z_{1N}(\jmath\omega)$ at ω_m. We can write

$$\left|\frac{\lambda_N}{\gamma_1}\overline{Z}_{1N}(\jmath\omega_\mathrm{m})\right|^2 = \frac{\tau + (\frac{\tau}{g_\mathrm{i}^2\gamma_1} - g_\mathrm{r})^2}{\tau + (\frac{\tau}{g_\mathrm{i}^2\lambda_N} - g_\mathrm{r})^2} \tag{C.4}$$

with $\tau = g_\mathrm{r}^2\sqrt{1 + \frac{2g_\mathrm{i}^2\lambda_1}{g_\mathrm{r}}} - g_\mathrm{r}^2$. We can approximate the square root as $\sqrt{1 + \frac{2g_\mathrm{i}^2\lambda_1}{g_\mathrm{r}}} = 1 + \frac{g_\mathrm{i}^2}{g_\mathrm{r}}\lambda_1$ to get $\tau \approx g_\mathrm{r}g_\mathrm{i}^2\lambda_1$. Then the modulus simplifies to

$$\left|\frac{\lambda_N}{\gamma_1}\overline{Z}_{1N}(\jmath\omega_m)\right|^2 \approx \frac{\left(\frac{\lambda_1}{\gamma_1} - 1\right)^2 + \frac{g_\mathrm{i}^2}{g_\mathrm{r}}\lambda_1}{\left(\frac{\lambda_1}{\lambda_N} - 1\right)^2 + \frac{g_\mathrm{i}^2}{g_\mathrm{r}}\lambda_1}. \tag{C.5}$$

Note that for N large, $\lambda_1 \to 0$ and $\lambda_N \to 4$. Then $\frac{g_\mathrm{i}^2}{g_\mathrm{r}}\lambda_1 \approx 0$ and $\left(\frac{\lambda_1}{\lambda_N} - 1\right)^2 + \frac{g_\mathrm{i}^2}{g_\mathrm{r}}\lambda_1 \approx 1$. We can write

$$\left|\frac{\lambda_N}{\gamma_1}\overline{Z}_{1N}(\jmath\omega_\mathrm{m})\right| \approx \left|\frac{\lambda_1}{\gamma_1}-1\right|. \tag{C.6}$$

Let us show that λ_1/γ_1 is bounded. γ_1 is an eigenvalue of a principal submatrix $\bar{L}_{NN}$ of L_p

$$\bar{L}_{NN} = \begin{bmatrix} 2 & -1 & \dots & 0 \\ -1 & \ddots & \ddots & \dots \\ \dots & \ddots & \ddots & -1 \\ \dots & 0 & -1 & 2 \end{bmatrix} \in \mathbb{R}^{N-1\times N-1}. \tag{C.7}$$

The eigenvalues for such a graph are as follows (Parlangeli and Notarstefano, 2012, Prop. 3.3)

$$\gamma_i = 2\left(\cos\frac{i\pi}{N}-1\right) = 4\sin^2\left(\frac{i\pi}{2N}\right). \tag{C.8}$$

Using $\sin x \geq 2x/\pi$ we get $\gamma_i \geq \frac{4}{N^2}$. Then using (4) $\lambda_1/\gamma_1 \leq \frac{4\pi^2}{(4N)^2}/\frac{4}{N^2} \leq \pi^2/16$. Then $|\lambda_1/\gamma_1 Z_{1N}(\jmath\omega_m)| > |1-\pi^2/16| = \zeta_1$, so it is bounded regardless of N. To conclude,

$$T_{N,N}(\jmath\omega_\mathrm{m}) \geq T_{N,N}(0)T_1(\jmath\omega_\mathrm{m})Z_{1N}(\jmath\omega_\mathrm{m}) \geq c_1N^2. \tag{C.9}$$

Now we show the upper bound. It follows from (C.6) that $|Z_{1N}(\jmath\omega_\mathrm{m})| \leq 1 = \zeta_2$. We now show that $\prod_{i=2}^{N-1}|Z_{ii}(\jmath\omega_\mathrm{m})| < \zeta_3$. Following the reasoning in (C.4), we can write

$$\left|\frac{\lambda_i}{\gamma_i}Z_{ii}(\jmath\omega_\mathrm{m})\right|^2 = \frac{\tau+\left(\frac{\tau}{g_\mathrm{i}^2\gamma_i}-g_\mathrm{r}\right)^2}{\tau+\left(\frac{\tau}{g_\mathrm{i}^2\lambda_i}-g_\mathrm{r}\right)^2} \approx \frac{\left(\frac{\lambda_1}{\gamma_i}-1\right)^2}{\left(\frac{\lambda_1}{\lambda_i}-1\right)^2} \tag{C.10}$$

$$\Rightarrow \left|\frac{\lambda_i}{\gamma_i}Z_{ii}(\jmath\omega_\mathrm{m})\right| = \frac{1-\frac{\lambda_1}{\gamma_i}}{1-\frac{\lambda_1}{\lambda_i}}. \tag{C.11}$$

We can bound the product $\prod_{i=2}^{N-1}\left|\frac{\lambda_i}{\gamma_i}Z_{ii}(\jmath\omega_\mathrm{m})\right|$ as

$$\begin{aligned}\prod_{i=2}^{N-1}\frac{\lambda_i}{\gamma_i}|Z_{ii}(\jmath\omega_\mathrm{m})| &= \frac{1-\frac{\lambda_1}{\gamma_2}}{1-\frac{\lambda_1}{\lambda_2}}\frac{1-\frac{\lambda_1}{\gamma_3}}{1-\frac{\lambda_1}{\lambda_3}}\dots\frac{1-\frac{\lambda_1}{\gamma_{N-1}}}{1-\frac{\lambda_1}{\lambda_{N-1}}} \\ &\leq \frac{1-\frac{\lambda_1}{\lambda_3}}{1-\frac{\lambda_1}{\lambda_2}}\frac{1-\frac{\lambda_1}{\lambda_4}}{1-\frac{\lambda_1}{\lambda_3}}\dots\frac{1-\frac{\lambda_1}{\gamma_{N-1}}}{1-\frac{\lambda_1}{\lambda_{N-1}}} = \frac{1-\frac{\lambda_1}{\gamma_{N-1}}}{1-\frac{\lambda_1}{\lambda_2}} \\ &\leq \frac{1}{1-\frac{\lambda_1}{\lambda_2}} \leq \frac{1}{1-\frac{\pi^2}{36}} = \zeta_3. \end{aligned} \tag{C.12}$$

We used the facts that $\lambda_{i+1} \geq \gamma_i$ and that $\frac{\lambda_1}{\lambda_2} \leq \frac{\pi^2}{36}$ using (B.5). It follows that

$$\begin{aligned}|T_{N,N}(\jmath\omega_\mathrm{m})| &= T_{N,N}(0)|T_1(\jmath\omega_\mathrm{m})||Z_{1N}(\jmath\omega_\mathrm{m})|\prod_{i=2}^{N-1}\frac{\lambda_i}{\gamma_i}|Z_{ii}(\jmath\omega_\mathrm{m})| \\ &\leq \zeta_2\zeta_3T_{N,N}(0)|T_1(\jmath\omega_\mathrm{m})| \leq c_2N^2. \end{aligned} \tag{C.13}$$

Combining (C.9) with (C.13) we get the quadratic scaling at ω_m. Note that $\|T_1(s)\|_\infty > \|T_i(s)\|_\infty$. That is why that at ω_m is not only the peak of $|T_1(\jmath\omega)|$, but also of $|T_{N,N}(\jmath\omega)|$. Thus, the bounds on the peak of $|T_{N,N}(\jmath\omega)|$ are the bounds on the $\mathcal{H}_\infty$ norm. □

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