Scaling in bidirectional platoons with dynamic controllers and proportional asymmetry

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***Abstract*—We consider platoons composed of identical vehicles with an asymmetric nearest-neighbor interaction. We restrict ourselves to intervehicular coupling realized with dynamic arbitrary-order onboard controllers such that the coupling to the immediately preceding vehicle is proportional to the coupling to the immediately following vehicle. Each vehicle is modeled using a transfer function and we impose no restriction on the order of the vehicle. The only requirement on the controller and vehicle model is that the platoon is stable for any number of vehicles. The platoon is described by a transfer function in a convenient product form. We investigate how the H-infinity norm and the steady-state gain of the platoon scale with the number of vehicles. We conclude that if the open-loop transfer function of the vehicle contains two or more integrators and the second smallest eigenvalue of the graph Laplacian is uniformly bounded from below, the norm scales exponentially with the growing distance in the graph. If there is just one integrator in the open loop, we give a condition under which the norm of the transfer function is bounded by its steady-state gain—the platoon is string-stable. Moreover, we argue that in this case it is always possible to design a controller for the extreme asymmetry—the predecessor following strategy.**

***Index Terms*—Vehicle platoon, string stability, asymmetric control, scaling, transfer functions.**

I. Introduction

Vehicle platoons are chains of automatic cars that are supposed to travel with a tight spacing in a highway lane. They are expected to increase the safety and capacity of highways. A number of theoretical results are available in the literature, but experiments with short vehicle platoons were described too [1] (PATH project) or [2] (SARTRE project). Majority of the practical results rely on intervehicular communication. The most commonly adopted approaches are *Cooperative Adaptive Cruise Control* (CACC) [1], [3], *leader following* [4] and *leader's velocity transmission* [5], [6]. However, the communication can be delayed, disturbed or even denied by an intruder.

In the absence of intervehicular communication, the only available information is the one measured by the onboard sensors, especially the intervehicular distances. It turns out that certain properties of such platoons need not scale well for a growing number of vehicles. Among the strategies, the *time-headway policy* is scalable [7] but the platoon's length grows with the speed of the leader. Among fixed-distance approaches such as the *predecessor following* and *symmetric or asymmetric bidirectional control*, an unpleasant phenomenon known as *string instability* can occur. This means that a disturbance affecting a given vehicle can be amplified as it propagates along the platoon (string) of vehicles. For the predecessor following strategy, string instability occurs for an arbitrary model of a vehicle as long as there are least two integrators in the open loop [8]. If measurements of the distance from both the immediately preceding and the immediately following vehicles are available, we call the corresponding control bidirectional. In this paper we are going to revolve around the role of asymmetry of bidirectional coupling.

Recent works suggest that in a bidirectional platoon with second-order open-loop dynamics, a good trade-off between the settling time and peaks in the transient response can be achieved if the asymmetry of coupling is imposed differently on the measured intervehicular distances and their first derivatives—relative velocities. However, these results are only obtained by numerical simulations [9] or the results are based on reasonable conjectures [10], [11]. Moreover, they are all valid only for particular system models. No general knowledge is available so far.

In contrast, if the coupling assumes identical asymmetry for both the distances and their first derivatives, a nonzero lower bound on the formation eigenvalues can be achieved [5]. This guarantees controllability [6] of the formation of an arbitrary size. On the other hand, for a double integrator model, the $\mathcal{H}_\infty$ norm of a particular transfer function related to disturbance attenuation grows exponentially in the number of vehicles [12]. Later this bad scaling was attributed to the presence of the uniform bound on eigenvalues if there are at least two integrators in the open loop [13]. Hence, the uniform boundedness of eigenvalues plausible from the perspective of faster transient response must be paid for by very bad scaling in the frequency domain.

If symmetric coupling is implemented, the norm grows only linearly [14], [15] but the step response suffers from very long transients—the eigenvalues get arbitrarily close to the origin. This can be alleviated using a wave-absorbing controller implemented on either end of the platoon [16]. Finally, it is also the sensitivity of the platoon to the noise that depends on the number of integrators in the open loop [17].

In this paper we consider platoons composed of identical vehicles with an asymmetric nearest-neighbor interaction. We restrict ourselves to the case when the coupling to the immediately preceding vehicle is proportional to the coupling to the immediately following vehicle (see eq. (1)). Each vehicle is modeled by a transfer function and we impose no restriction on the order or structure of the model.

We investigate how the $\mathcal{H}_\infty$ norm and the steady-state gain of the platoon scale with the number of vehicles. If the vehicle contains two or more integrators and the eigenvalues of the graph Laplacian are uniformly bounded from below, the norm scales exponentially with the growing distance in the graph (Sec. IV-A). If there is just one integrator in the open loop, we give a condition under which the norm of the transfer function is bounded by its steady-state gain—the platoon is string-stable (Sec. IV-B). In addition, in this case it is possible to design a string-stable controller for the extreme asymmetry—the predecessor following strategy, which offers some implementation advantages compared to general asymmetric bidirectional control (see Sec. IV-C).

The novelty is that our results hold for an *arbitrary LTI model* (order and structure) of the individual vehicle. Thus, we do not limit ourselves to a single or double integrator models as in [5], [6], [9], [12], [17]. In fact, our work generalizes those results to arbitrary transfer function models of individual vehicles. The main distinguishing feature is the number of integrators in the open loop. We extend the result on exponential scaling from our paper [13] to an arbitrary transfer function in the formation. Moreover, we add a discussion of scaling when only one integrator in the open loop is present in the agent model and also a steady-state gain is analyzed.

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This paper therefore should give a broader qualitative overview of what is achievable with proportional asymmetry for general vehicle models.

II. Vehicle and platoon modelling

Consider N *identical* vehicles indexed as $i = 1, 2, \dots, N$, with $i = 1$ corresponding to the platoon leader. The leader drives independently of the platoon. The vehicles have identical transfer functions $G(s) = \frac{b(s)}{a(s)}$ of an arbitrary type and order with positions y_i as the outputs. Given the transfer function of the vehicle, a dynamic controller $R(s)$ for the vehicle is designed in order to meet some platoon requirements, such as to stabilize the platoon (see Assumption 1) or satisfy the internal model principle [18]. This controller produces the input to the vehicle and we assume that it is of an arbitrary structure and order, provided that the requirements on the platoon are satisfied. The open-loop model $M(s) = R(s)G(s) = \frac{b(s)q(s)}{a(s)p(s)}$ is a series connection of the controller and the vehicle models.

Definition 1 (Number of integrators in the open loop)**.** Let the open-loop model be factored as $M(s) = 1/s^\eta\ \overline{M}(s)$ with $\overline{M}(0) < \infty$. Then $\eta \in \mathbb{N}_0$ is the number of integrators in the open loop.

The number η is also known as a *type number* of the system. For instance, the model $M(s) = \frac{1}{s(s+a)}$ is a system with one integrator in the open loop and $M(s) = \frac{s+1}{s^2(s+b)}$ has $\eta = 2$. We call the well-known cases with $\overline{M}(s) = 1$ a single-integrator system for $\eta = 1$ and a double-integrator system for $\eta = 2$, respectively.

The input to the controller is the combined front and rear intervehicular spacing error

$$e_i = (y_{i-1} - y_i) - \epsilon_i(y_i - y_{i+1}) + r_i. \tag{1}$$

We call the nonnegative weight ϵ_i of the rear spacing error the *constant of bidirectionality*. When $\epsilon_i = 1$, $\forall i$, we have a symmetric control, when $\epsilon_i \neq 1$, $\forall i$ the control is asymmetric and $\epsilon_i = 0, \forall i$ we have the predecessor following (PF). The general external input r_i can represent, for instance, a measurement noise or a reference such as the reference distance d_{ref}. In such a case $r_i = -d_{\text{ref}} + \epsilon_i d_{\text{ref}}$ and the distances $\Delta_i = y_{i-1} - y_i$ are regulated to d_{ref}. The leader's control input is just r_1 and the controller of the trailing vehicle has the input $e_N = (y_{N-1} - y_N) + r_N$. Since we use a dynamic controller, the control law can also access the relative velocity and other derivatives of the distances (for instance by using a PD controller $R(s) = \alpha s + \beta$).

A. Laplacian properties

The regulation errors in (1) are given in a vector form as $e = -Ly + r$ with $e = [e_1, \dots, e_N]^T, y = [y_1, \dots, y_N]^T$ and $r = [r_1, \dots, r_N]^T$. The matrix $L = [l_{ij}] \in \mathbb{R}^{N \times N}$ is the Laplacian of a path graph and has the following structure

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ -1 & 1+\epsilon_2 & -\epsilon_2 & 0 & \dots \\ 0 & -1 & 1+\epsilon_3 & -\epsilon_3 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & -1 & 1+\epsilon_{N-1} & -\epsilon_{N-1} \\ 0 & \dots & 0 & -1 & 1 \end{bmatrix}. \tag{2}$$

It is a non-symmetric tridiagonal matrix. Next we state some useful properties of L, mainly taken from the literature.

Lemma 1. *Laplacian L in (2) and its eigenvalues λ_i have the following properties:*

a) *The eigenvalues λ_i are all real and $\lambda_i \geq 0$, $\forall i$.*
b) *With the eigenvalues ordered as $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$, the smallest eigenvalue $\lambda_1 = 0$ and this eigenvalue is simple.*
c) *The eigenvalues are upper-bounded by $\lambda_{\max}$, that is, $\lambda_i \leq \lambda_{\max} \leq 2 \max_i(1 + \epsilon_i)$.*
d) *Let L_r be the matrix obtained from L by deleting the first row and the first column (both correspond to the leader). Then $\lambda_i(L) = \lambda_i(L_r)$ for all $\lambda_i \neq 0$.*
e) *Suppose that $\epsilon_i \leq \epsilon_{\max} < 1\,\forall i$. Then the nonzero eigenvalues $\lambda_2, \dots, \lambda_N$ are upper-bounded by $\lambda_i \leq \lambda_{\max} = 2(1 + \epsilon_{\max}), \forall i \geq 1$ and lower-bounded by*

$$\lambda_i \geq \lambda_{\min} \geq \frac{1}{2}\frac{(1-\epsilon_{\max})^2}{1+\epsilon_{\max}} > 0, \qquad \forall i \geq 2. \tag{3}$$

The bounds are uniform, that is, they do not depend on N.
f) *Let L_k be a matrix obtained from L by deleting kth row and column. Let the eigenvalues of L_k, $1 < k < n$, be $\mu_1 < \mu_2 < \dots < \mu_{n-1}$. Then*

$$\lambda_{j+2} \geq \mu_j \geq \lambda_j,\ j = 1, 2, \dots, N-2. \tag{4}$$

Proof. The properties a)-d) are discussed in [13, Lem. 1], e) is proved in [13, Thm. 1]. The statement f) follows from [19, Thm. 5.5.6], which gives conditions of interlacing for totally nonnegative matrices. L is similar to a totally nonnegative matrix [19, pp. 6,7]. Both L and L_k can be transformed to totally nonnegative matrices using similarity transform with signature matrices $S = \text{diag}[1, -1, \dots, 1, -1]$. The results are $|L|$ and $|L_k|$ with the absolute values taken element-wise. Since $|L_k|$ is a principal submatrix of $|L|$, interlacing occurs. Since L is similar to $|L|$ and L_k to $|L_k|$, their eigenvalues interlace. $\square$

The property e) is an instance of uniform boundedness—the lower bound on eigenvalues $\lambda_{\min} > 0$ does not depend on N [5], [12], [13]. Applying f) repeatedly, the interlacing holds for any principal submatrix.

Remark 1. In [13] we considered a more general model with a different controller weight μ_i for each vehicle such that $L_\mu = WL$, $W = \text{diag}[\mu_1, \mu_2, \dots, \mu_N]$. For the clarity of presentation we restricted ourselves here to L in (2) and $\mu_i = 1\,\forall i$, although all the results (apart from the steady-state gain) would remain unchanged.

B. Transfer functions

We are interested in how the vector of external inputs r (acting at the inputs of the controller) affects the vector of positions y of vehicles. This is in general described by a transfer function matrix $y(s) = \mathbf{T}(s)r(s)$. The (O, C)th element of matrix $\mathbf{T}(s)$ is denoted by $T_{CO}(s) = \frac{y_O(s)}{r_C(s)}$, $C = 1, \dots, N, O = 1, \dots, N$. The transfer function $T_{CO}(s)$ therefore describes the effect of the external input r_C acting at a vehicle indexed C (called a *control vehicle*) on the position y_O of the vehicle with an index O (called an *output vehicle*)—see Fig. 1. We will be interested in how its $\mathcal{H}_\infty$ norm defined as $\|T_{CO}(s)\|_\infty = \sup_{\omega \geq 0} |T_{CO}(\jmath\omega)|$ scales with a growing number N of vehicles and the distance d_{CO} in a graph. We use the statement "from C to O" with the meaning of "from the input r_C of the vehicle C to the output y_O of the vehicle O". The indices C and O can be chosen arbitrarily. Note that due to bidirectional

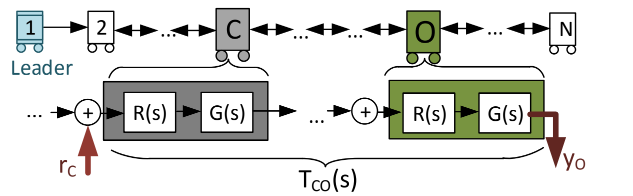


Fig. 1: Block diagram showing the transfer function $T_{CO}(s)$.

architecture, for any selection of C, O the transfer function $T_{CO}(s)$ depends on the whole formation.

Since the graph of a platoon is a path graph, there is only one directed path from the node C to the node O. This path is a sequence of edges with the weights $w_{i,j}$. The weight of the path is $w_{CO} = \prod_{j=C}^{O-1} w_{j,j+1}$. In our case $w_{i,i+1} = 1$ and $w_{i+1,i} = \epsilon_i$, so

$$w_{CO} = \begin{cases} 1 & \text{for } C \le O, \\ \prod_{i=O}^{C-1} \epsilon_i & \text{for } C > O. \end{cases} \tag{5}$$

The number of edges on the directed path from the node C to the node O is called the graph distance d_{CO} between C and O. We use the following product form of $T_{CO}(s)$ that we derived in [20, Thm. 5]

$$T_{CO}(s) = w_{CO} \frac{[b(s)q(s)]^{d_{CO}+1} \prod_{i=1}^{N-d_{CO}-1} [a(s)p(s) + \gamma_i b(s)q(s)]}{\prod_{i=j}^{N} [a(s)p(s) + \lambda_j b(s)q(s)]}, \tag{6}$$

where λ_j is the jth eigenvalue of L. The coefficients $\gamma_i \in \mathbb{R}$, $\gamma_i \le \gamma_{i+1}$, are the eigenvalues of the matrix $\hat{L} \in \mathbb{R}^{N-d_{CO}-1 \times N-d_{CO}-1}$ that is obtained from L by deleting all the rows and columns corresponding to the nodes on the path from C to O, see [20, Thm. 10]. Note that $\hat{L}$ is a principal submatrix of L, hence interlacing in the sense of Lemma 1 f) holds. For instance, for a formation with $C = 3$, $O = 4$ and $N = 5$, we delete the third and the fourth rows and columns of L to get $\hat{L}$ with the eigenvalues $\gamma_i = [0, 1, 1 + \epsilon_2]$,

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 1+\epsilon_2 & -\epsilon_2 & 0 & 0 \\ 0 & -1 & 1+\epsilon_3 & -\epsilon_3 & 0 \\ 0 & 0 & -1 & 1+\epsilon_4 & -\epsilon_4 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \Longrightarrow \hat{L} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1+\epsilon_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{7}$$

Using the statement d) in Lemma 1, we can exclude the leader from the formation (and also get rid of $\lambda_1 = 0$ and $\gamma_1 = 0$). Whenever we analyze a transfer-function norm, we will work with L_r and all the indices will start from 2. This means that we use so called pinning control scheme [21]. The leader can be included afterwards by multiplying the transfer function $T_{CO}(s)$ by $M(s)$.

Assumption 1. The overall platoon with the Laplacian L_r is asymptotically stable for any N.

In other words, for a given vehicle model $G(s)$, we assume that the controller $R(s)$ was designed in such a way that the platoon is stabilized for any number of vehicles. This is a necessary condition for a $\mathcal{H}_\infty$ norm analysis. The cases when there exists no controller which stabilizes the platoon for any N are not considered in this paper.

In order to achieve stability, it follows (similarly to [22]) from (6) that the polynomial $a(s)p(s) + \lambda_j b(s)q(s)$ must be stable for any $\lambda_j \in [\lambda_{\min}, \lambda_{\max}]$, $\lambda_j \in \mathbb{R}$. Since by the assumption the stability is required for L_r, the term corresponding to $\lambda_1 = 0$ was excluded (the leader was excluded from the formation). Note that $a(s)p(s) + \lambda_j b(s)q(s)$ is a standard form for the denominator in the root-locus theory for the system $\lambda_j M(s)$ with the gain λ_j. Thus, it is necessary to stabilize the single-agent system $\lambda_j M(s)$ for a bounded interval of the real gain $\lambda_j \in [\lambda_{\min}, \lambda_{\max}]$. From (4) it follows that also $\gamma_i \in [\lambda_{\min}, \lambda_{\max}]$, $\forall i$, so if the system is asymptotically stable, all its zeros are in the left half-plane too.

III. Steady-state gain of transfer functions

Besides the $\mathcal{H}_\infty$ norm, another important control-related characteristic of a platoon is the steady-state gain $T_{CO}(0)$. By the internal model principle [18] we assume that $\eta \ge 1$ to enable the vehicles to track the leader's constant velocity. With at least one integrator in $M(s)$ we get $a(0)p(0) = 0$. After excluding the leader, the steady-state gain follows from (6) as

$$\begin{aligned} T_{CO}(0) &= w_{CO} \frac{[b(0)q(0)]^{d_{CO}+1} \prod_{i=2}^{N-d_{CO}-1} [\gamma_i b(0)q(0)]}{\prod_{j=2}^{N} [\lambda_j b(0)q(0)]} \\ &= w_{CO} \frac{\prod_{i=2}^{N-d_{CO}-1} \gamma_i}{\prod_{j=2}^{N} \lambda_j}. \end{aligned} \tag{8}$$

This shows that the steady-state gain *does not depend on the dynamic model* of an individual agent, it is only a function of the structure of the network (λ_j and γ_i are both obtained from L). We can now apply the previous result to get the steady-state gain of the transfer function $T_{CO}(s)$ in vehicle platoons.

Theorem 1. *The steady-state gain of the platoon is given by*

$$T_{CO}(0) = \begin{cases} w_{CO} \left(1 + \sum_{i=1}^{C-2} \prod_{j=1}^{i} \epsilon_{C-j}\right) & \text{for } C \le O \\ w_{CO} \left(1 + \sum_{i=1}^{O-2} \prod_{j=1}^{i} \epsilon_{O-j}\right) & \text{for } O < C \end{cases} \tag{9}$$

The proof is in Appendix A. Note that for $C \le O$, the steady-state gain does not depend on O as $w_{CO} = 1$ for $C \le O$. We can discuss several cases relevant for the platoon control.

Corollary 1. *If there is $\epsilon_{\max}$ such that $\epsilon_i \le \epsilon_{\max} < 1$ $\forall i$, then $T_{CO}(0)$ is upper bounded as $T_{CO}(0) \le \frac{1}{1-\epsilon_{\max}}$. This holds for all N and for all C, O.*

Proof. We can bound the product in (9) as $\prod_{j=1}^{i} \epsilon_{C-j} \le \epsilon_{\max}^i$. Then $T_{CO}(0) \le w_{CO} \left(1 + \sum_{i=1}^{C-2} \epsilon_{\max}^i\right) \le w_{CO} \frac{1}{1-\epsilon_{\max}}$, since $\sum_{i=0}^{\infty} \epsilon_{\max}^i = \frac{1}{1-\epsilon_{\max}}$. The same holds for $w_{CO} \left(1 + \sum_{i=1}^{O-2} \prod_{j=1}^{i} \epsilon_{O-j}\right) \le w_{CO} \frac{1}{1-\epsilon_{\max}}$. If $C \le O$, then $w_{CO} = 1$. If $C > O$, then $w_{CO} = \prod_{i=C-1}^{O} \epsilon_i \le \epsilon_{\max}^{d_{CO}} < 1$. Therefore, $T_{CO}(0) \le w_{CO} \frac{1}{1-\epsilon_{\max}} \le \frac{1}{1-\epsilon_{\max}}$. □

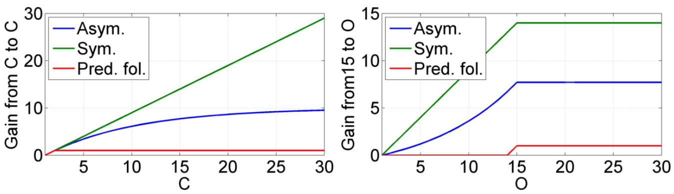
The bound on $T_{CO}(0)$ for the predecessor-following control strategy is one (note $\epsilon_{\max} = 0$), which is the minimum amidst all control strategies. For the symmetric bidirectional control we use (9) to get the steady-state gain equal to $C-1$, which shows that it is unbounded in N. This can be explained by the fact that all the vehicles ahead of the vehicle C have to increase the distance to neighbors by one. The steady-state gains for a fixed control node and a varying output node for several strategies are in Fig. 2b, while the gain from C to C is in Fig. 2a. Although the gain grows with C, for a fixed C, it does not grow with the number N of agents.

One might also be interested in the change of the intervehicular distance $\Delta_O = y_{O-1} - y_O$ as an effect of the input r_C. Then $T_\Delta(s) = \frac{\Delta_O(s)}{r_C(s)} = T_{C,O-1}(s) - T_{C,O}(s)$. Using (9) and (5), its steady-state gain is $T_\Delta(0) = 0$ for $O \ge C$ and $T_\Delta(0) = -\prod_{i=O}^{C-1} \epsilon_i$ for $O \le C$. This means that all the vehicles ahead of C have to increase their steady-state distances (unless $\epsilon_i = 0$, $\forall i$), while distances of the cars behind C remain unchanged. In asymmetric control with $\epsilon_i \le \epsilon_{\max} < 1 \forall i$ the change in distance will be less than one since $\prod_{i=O}^{C-1} \epsilon_i < 1$.

IV. Scaling of $\mathcal{H}_\infty$ norms in platoons

In this section we investigate how the $\mathcal{H}_\infty$ norm of an arbitrary transfer function $T_{CO}(s)$ changes when more vehicles are added (N grows). Define two types of transfer functions

$$T_j(s) = \frac{\lambda_j b(s)q(s)}{a(s)p(s) + \lambda_j b(s)q(s)}, \; Z_{ij}(s) = \frac{a(s)p(s) + \gamma_i b(s)q(s)}{a(s)p(s) + \lambda_j b(s)q(s)}. \tag{10}$$



(a) Varying C and O, $O = C$. (b) Fixed $C = 15$, varying O.

Fig. 2: Steady-state gains for different choices of C and O and for $\epsilon = 0.9$ (Asym.), $\epsilon = 1$ (Sym.) and $\epsilon = 0$ (Pred. fol).

From the product (6), we can form $d_{CO} + 1$ transfer functions of type $T_j(s)$ and $N - d_{CO} - 1$ of type $Z_{ij}(s)$, up to the gain. Let $T_{\min}(s)$ be the transfer function of the closed-loop system

$$T_{\min}(s) = \frac{\lambda_{\min} b(s) q(s)}{a(s)p(s) + \lambda_{\min} b(s) q(s)} \tag{11}$$

with $\lambda_{\min}$ acting as a proportional gain ($\lambda_{\min} > 0$ is the lower bound on $\lambda_i, i \geq 2$) Similarly, for the upper bound on eigenvalues $\lambda_{\max}$ let $T_{\max}(s)$ be the corresponding closed loop. Note that $|T_j(0)| = 1$ due to at least one integrator in the open loop, hence $\|T_j(s)\|_\infty \geq 1$. The next technical Lemma is proved in Appendix B.

Lemma 2. *Let $\lambda_j M(\jmath\omega_0) = \alpha_j + \jmath\beta_j$ for some frequency $\omega_0 > 0$, $\alpha_j, \beta_j \in \mathbb{R}$, $\jmath = \sqrt{-1}$. Then*

a) *If $|T_i(\jmath\omega_0)| > 1$, then $|T_j(\jmath\omega_0)| > 1 \quad \forall \lambda_j \geq \lambda_i$ and $\alpha_j < -1/2$.*
b) *If $|T_i(\jmath\omega_0)| \leq 1$, then $|T_j(\jmath\omega_0)| \leq 1 \quad \forall \lambda_j \leq \lambda_i$ and $\alpha_j \geq -1/2$.*
c) $|Z_{ij}(\jmath\omega_0)| \geq |Z_{ij}(0)|$ *for* $\{\alpha_j \leq -1$ *and* $\gamma_i \geq \lambda_j\}$
d) $|Z_{ij}(\jmath\omega_0)| \geq |Z_{ij}(0)|$ *for* $\{-1 < \alpha_j \leq -\frac{1}{2}$ *and* $\gamma_i \leq \lambda_j\}$
e) $|Z_{ij}(\jmath\omega_0)| \leq |Z_{ij}(0)|$ *for* $\{\alpha_j > -\frac{1}{2}$ *and* $\gamma_i \geq \lambda_j\}$

A. Exponential growth

It was proved in [13] that the response of the last vehicle grows exponentially in N due to the presence of a uniform nonzero lower bound on the eigenvalues. However, the analysis was done only for one transfer function in the platoon and one input—the movement of the leader. The next theorem proved in Appendix C extends the exponential scaling to an arbitrary transfer function in a finite platoon. The test involves only the closed-loop $T_{\min}(s)$ of an individual agent.

Theorem 2. *If $\|T_{\min}(s)\|_\infty > 1$ and the eigenvalues of L are uniformly bounded from zero, then there are two real constants $0 < \xi \leq 1$ and $\zeta > 1$ depending only on $\lambda_{\min}, \lambda_{\max}$ and $M(s)$ such that $\|T_{CO}(s)\|_\infty > \zeta^{d_{CO}}\, T_{CO}(0)\, \xi^2$. That is, the norm $\|T_{CO}(s)\|_\infty$ grows exponentially with the graph distance d_{CO}.*

The effect of the input r_C applied at the control node gets exponentially amplified with the graph distance between C and O. Hence, it is amplified as it propagates further from the control node even in a platoon with fixed N. Figure 3 shows scaling for a third-order model with varying asymmetry in a given range. If $O < C$, then $T_{CO}(0)$ given in (9) might decrease faster than $\zeta^{d_{CO}}$ grows and the norm might be less than one (Fig. 3c). If $C \leq O$, then $\|T_{CO}(s)\|_\infty \gg 1$ for large d_{CO} (Fig. 3a). In Fig. 3b we show how $\|T_{CO}(s)\|_\infty$ changes with a graph distance — $C = 3$ is kept fixed and O is varied, so that d_{CO} grows with growing O.

Two integrators in the open loop ($\eta = 2$) are necessary for tracking of the leader moving with a constant velocity [23, Lem. 3.1]. However, for at least two integrators in the open-loop we have $\|T_{\min}(s)\|_\infty > 1$ [8, Thm. 1]. For a Laplacian with uniformly bounded eigenvalues this means that $\|T_{CO}(s)\|_\infty$ grows exponentially with the distance d_{CO} and there is no linear controller which could prevent this. Thus, we cannot have a good behavior with a uniform bound and two integrators. The main results of [8], [12], [13] are special cases of Theorem 2, since asymmetric Laplacian with $\epsilon_i \leq \epsilon_{\max} < 1$ has uniformly bounded eigenvalues, see Lemma 1 e). Nevertheless, even a platoon with $\eta = 1$ can exhibit exponential scaling.

B. Design of a string stable controller

So far we have discussed situations in which the system scales badly. In this section we provide a test for the string stability. One of the most common string stability conditions in vehicle platoons is $\left\| \frac{y_i(s)}{y_{i-1}(s)} \right\|_\infty \leq 1 \quad \forall i$, used e.g., in [1] (see [24] for other definitions of string stability). In other words, the effect of disturbance at one vehicle must be attenuated when propagated along the platoon. However, in a bidirectional platoon the signal can propagate in both directions.

Definition 2 (Bidirectional string stability)**.** The bidirectional platoon is *string-stable* if for an input r_C acting at vehicle C the output y_O at vehicle O satisfies

$$\left\| \frac{y_O(s)}{y_{O-1}(s)} \right\|_\infty \leq 1, \forall O \geq C; \; \left\| \frac{y_{O-1}(s)}{y_O(s)} \right\|_\infty \leq 1, \forall O < C. \tag{12}$$

We can now state a very simple sufficient condition for the bidirectional string stability, again involving only a norm of the closed loop of an individual agent. The proof is in Appendix D.

Theorem 3. *If $\|T_{\max}(s)\|_\infty = 1$, then $\|T_{CO}(s)\|_\infty = |T_{CO}(0)|$ and the platoon is bidirectionally string-stable.*

The first part states that the $\mathcal{H}_\infty$ norm of $T_{CO}(s)$ equals its steady-state gain (which is only a function of the interconnection structure). If the upper bound $\lambda_{\max}$ on eigenvalues of L_{r} is independent of N, the bidirectional string stability holds for all N, all ϵ_i and for every $T_{CO}(s)$.

The condition $\|T_{\max}(s)\|_\infty = 1$ provides a simple way how to tune a SISO controller for a vehicle model $G(s)$ in a platoon of arbitrary size. To achieve $\|T_{\max}(s)\|_\infty = 1$, there must be at most one integrator in the open loop. Systems with one integrator in the open loop were used in [6], [25], despite the fact that they cannot track the leader's position. This is usually overcome using leader's velocity as the reference velocity. However, this is a *centralized information* and the leader's velocity needs to be broadcast perpetually, which requires a communications infrastructure.

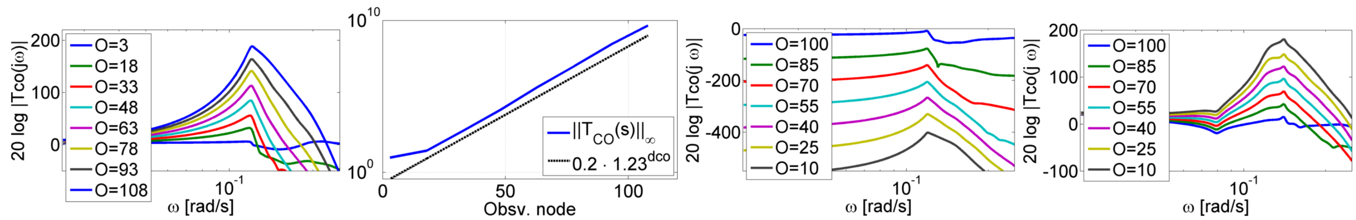
C. Design of a predecessor following controller

For a platoon with uniformly bounded eigenvalues it follows from Theorem 2 that $\|T_{\min}\|_\infty = 1$ is necessary for string stability. Denote a standard closed-loop as $T(s) = M(s)/(1 + M(s))$.

Lemma 3. *If there is a bidirectionally-string-stable asymmetric control for a given $G(s)$, then there always exists a predecessor following controller ($\epsilon = 0$) achieving $\|T(s)\|_\infty = 1$.*

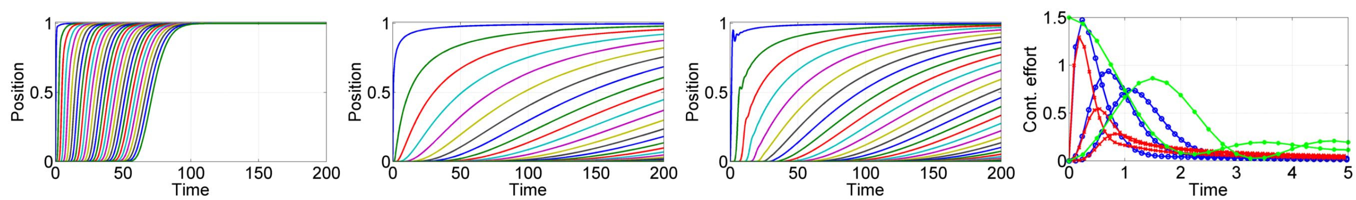
As an example of the closed loop, take $T(s) = T_{\min}(s)$ since $\|T_{\min}(s)\|_\infty = 1$—the gain of the controller was just decreased to $\lambda_{\min}$. Since such a system might have a slow transient response, the controller can be redesigned.

The simulation results are in Fig. 4. We designed two controllers for the system model $G(s) = \frac{1}{s^2+0.5s}$. The controller $R_1(s) = \frac{2.4s+1}{0.125s+1}$ achieves $\|T(s)\|_\infty = 1$ for predecessor following (PF). In addition to that, it also has a positive impulse response, which is very useful in platoon control. Both properties together guarantee string stability for PF in $\mathcal{L}_\infty$-induced norm [26]. The necessary



(a) $C=3$, $O \geq C$, $\epsilon_i \in [0.4, 0.6]$ (b) $\|T_{CO}(s)\|_\infty$ for a), $C=3$ (c) $C=105$, $O \leq C$, $\epsilon_i \in [0.4, 0.6]$ (d) $C=105$, $C \geq O$, $\epsilon_i \in [1.4, 1.6]$

Fig. 3: Scaling of $|T_{CO}(\jmath\omega)|$ as a function of C kept fixed and O varying with $N=110$. The model is a PI controller $R=\frac{s+1}{s}$ designed for a vehicle model $G=\frac{1}{s^2+5s}$, hence $\eta=2$ and the vehicle can track the leader moving with constant velocity. ϵ_i were randomly generated in the given range. Fig. 3b shows $\|T_{CO}(s)\|_\infty$ for the pairs C, O used in a) in semilog. coordinates. It is clear that the norm scales exponentially.



(a) Pred.fol. with $R_1(s)$ (b) Asym., $R_1(s)$, $\epsilon=0.9$ (c) Asym., $R_2(s)$, $\epsilon=0.9$ (d) Control effort for vehicles 2,3,4

Fig. 4: Responses to leader's step in position for different architectures for $N=150$. In 4d: blue - predecessor following., red - asymmetric control with $R_1(s)$, green - asymmetric control with $R_2(s)$.

conditions for positive response are dominant real pole and no real zero right from this pole [27]. The controller $R_2(s)=1.5$ is a simple proportional controller. A controller with a lower gain was used in [6]. It is apparent from Fig. 4 that for approximately the same maximal control effort, the PF achieves the best transient response among the cases shown.

Although in general we cannot guarantee better transients of PF compared to asymmetric bidirectional control, we think that PF offers many advantages: 1) no need for a rear-distance sensor, 2) developed theory for a closed-loop controller design (e. g., $\mathcal{H}_\infty$ approach), 3) easier handling of heterogeneity, 4) faster convergence time for the same maximal control effort—with the same controller the PF has a larger spectral gap (larger $\lambda_{\min}$). The performance could then be compared by simulations. Note that although the PF can have a better transient, a bidirectional architecture might still be required, e.g., for safety reasons. Then Theorem 3 gives a condition for design.

V. Conclusion

We investigated asymmetric control of vehicle platoons where proportional asymmetry is used—the front spacing error is proportional to the rear spacing error. First we analyzed scaling of steady-state gain of an arbitrary transfer function in a platoon. It was proved that it grows without bound with N for a symmetric bidirectional control scheme, while it stays bounded in a presence of asymmetry. We proved that for more than one integrator in the open loop, the asymmetric bidirectional control is not scalable, because the $\mathcal{H}_\infty$ norm of any transfer function grows exponentially with the graph distance. If we allow the vehicles to know the leader's velocity (which requires permanent communication), only one integrator in the open loop can be present. Then we provide a simple design method for tuning the controller to achieve bidirectional string stability. In this case also a string-stable predecessor following controller can always be designed. This paper thus gave an overview of the achievable performance in bidirectional control with proportional asymmetry.

Appendix A

Proof of Theorem 1. As stated in Sec. II.B., we will work with $L_{\rm r}=[l_{ij}]$. We begin by calculating the product in the denominator of (8). The product of all λ_i's equals $\det L_{\rm r}$. The recursive rule to calculate the determinant of tridiagonal matrix is [28, Lem. 0.9.10] $D_n = l_{n,n}D_{n-1} - l_{n,n+1}l_{n+1,n}D_{n-2}$, where D_n is the determinant of the submatrix of size n. We begin from bottom right corner of $L_{\rm r}$. Then $D_1=1$ (the bottom right element) and $D_2=1$. Then D_3 can be calculated as $D_3=(1+\epsilon_{N-2})D_2-\epsilon_{N-2}D_1=1$. By induction, the determinant of $L_{\rm r}$ is $\det L_{\rm r}=\prod_{j=2}^{N}\lambda_j=1$ for any size of $L_{\rm r}$.

Now we calculate the product in the numerator of (8). It equals the determinant of $\hat{L}$. Suppose that $C \leq O$. If $O < C$, then the indices C and O are swapped and only the weight of the path is different. The matrix $\hat{L}$ reads $\hat{L}=\mathrm{diag}(L_1, L_2)$ with

$$L_1 = \begin{bmatrix} 1+\epsilon_2 & -\epsilon_2 & 0 & .. & 0 \\ -1 & 1+\epsilon_3 & -\epsilon_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & -1 & 1+\epsilon_{C-1} \end{bmatrix}. \tag{13}$$

The matrix L_2 has the same structure as $L_{\rm r}$, hence $\det L_2 = 1$. The dimensions are $L_1 \in \mathbb{R}^{(C-2)\times(C-2)}$ and $L_2 \in \mathbb{R}^{(N-O-1)\times(N-O-1)}$.

The determinant of L_1 of size $n\times n$ can be recursively calculated as $\det L_{1,n} = (1+\epsilon_n)\det L_{1,n-1} - \epsilon_n \det L_{1,n-2}$. Let us start from the bottom right corner again. Then $\det L_{1,1} = 1+\epsilon_{C-1}$ and $\det L_{1,2} = 1+\epsilon_{C-1}+\epsilon_{C-1}\epsilon_{C-2}$. The determinant

$$\begin{aligned} \det L_{1,3} &= (1+\epsilon_{C-3})\det L_{1,2} - \epsilon_{C-3}\det L_{1,1} \\ &= 1+\epsilon_{C-1}+\epsilon_{C-1}\epsilon_{C-2}+\epsilon_{C-1}\epsilon_{C-2}\epsilon_{C-3}. \end{aligned} \tag{14}$$

The pattern is now apparent and the determinant of L_1 is $\det L_1 = 1+\sum_{i=1}^{C-2}\prod_{j=1}^{i}\epsilon_{C-j}$. The sum goes from 1 to $C-2$ because we excluded the leader from the formation and the vehicle C is part of the path from C to O, so $C-2$ vehicles remain. Since $\det \hat{L} = \det L_1 \det L_2$, the steady state gain is then $T_{CO}(0) = w_{CO}\frac{\det L_1 \det L_2}{\det L_{\rm r}} = w_{CO}\left(1+\sum_{i=1}^{C-2}\prod_{j=1}^{i}\epsilon_{C-j}\right).$ $\square$

Appendix B

Proof of Lemma 2. Proof of a): The proof can be found as a part of the proof of [13, Thm. 3]. It also follows from the proof that $|T_i(\jmath\omega_0)| > 1 \Leftrightarrow \alpha < -1/2$.

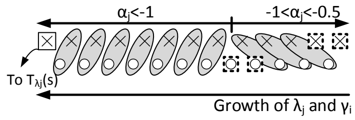


Fig. 5: Matching of λ_j and γ_i to form $Z_{ij}(s)$. Dashed pairs are the two $Z_{ij}(s)$ for which $|Z_{ij}(\jmath\omega_0)| > 1$ is not guaranteed.

Proof of b) follows from a). Suppose that $|T_j(\jmath\omega_0)| > 1$ for $\lambda_j < \lambda_i$. Then by a) also $|T_i(\jmath\omega_0)| > 1$, which contradicts the assumption $|T_i(\jmath\omega)| \leq 1$. Hence, $|T_j(\jmath\omega_0)| \leq 1$.

Proof of statements c)-e): The transfer function $Z_{ij}(s)$ can be written as $Z_{ij}(s) = \frac{1+\gamma_i M(s)}{1+\lambda_j M(s)}$. Its squared modulus at ω_0 is using $\kappa_{ij} = \frac{\gamma_i}{\lambda_j}$ given as

$$\begin{aligned}|Z_{ij}(\jmath\omega_0)|^2 &= \left|\frac{1+\kappa_{ij}(\alpha_j + \jmath\beta_j)}{1+(\alpha_j+\jmath\beta_j)}\right|^2 \\ &= \kappa_{ij}^2\left[1+\frac{\left(\frac{1}{\kappa_{ij}}-1\right)\left(2\alpha_j+1+\frac{1}{\kappa_{ij}}\right)}{(\alpha_j+1)^2+\beta_j^2}\right]. \end{aligned} \tag{15}$$

Denote the numerator $m_{ij} = \left(\frac{1}{\kappa_{ij}}-1\right)\left(2\alpha_j+1+\frac{1}{\kappa_{ij}}\right)$. The square of the steady-state gain is $|Z_{ij}(0)|^2 = \kappa_{ij}^2$. If $m_{ij} > 0$, then $|Z_{ij}(\jmath\omega_0)|^2 > |Z_{ij}(0)|^2 = \kappa_{ij}^2$ since $(\alpha_j+1)^2+\beta_j^2 > 0$. If $m_{ij} \leq 0$, then $|Z_{ij}(\jmath\omega_0)|^2 \leq |Z_{ij}(0)|^2$. Let us analyze the statements c)-e).

c) If $\alpha_j \leq -1$ and $\gamma_i \geq \lambda_j$, then $\left(\frac{1}{\kappa_{ij}}-1\right) \leq 0$ and also $\left(2\alpha_j+1+\frac{1}{\kappa_{ij}}\right) \leq 0$, hence $m_{ij} \geq 0$ which proves the statement c). d) If $-1 < \alpha_j \leq -\frac{1}{2}$ and $\gamma_i \leq \lambda_j$, so $\kappa_{ij} \leq 1$, then $\left(\frac{1}{\kappa_{ij}}-1\right) \geq 0$ and also $\left(2\alpha_j+1+\frac{1}{\kappa_{ij}}\right) > 0$, $m_{ij} \geq 0$ and d) is proved. e) If $\alpha_j > -\frac{1}{2}$ and $\gamma_i \geq \lambda_j$, then $\left(\frac{1}{\kappa_{ij}}-1\right) \leq 0$ and $\left(2\alpha_j+1+\frac{1}{\kappa_{ij}}\right) \geq 0$, hence $m_{ij} \leq 0$ and e) is proved. □

Appendix C

Proof of Theorem 2. In the proof we work with reduced Laplacian L_r. Let ω_0 be a frequency at which $|T_{\min}(\jmath\omega_0)| > 1$. The key idea is to form $T_j(s)$ and $Z_{ij}(s)$ from (6) as follows:

1) Take each term $a(s)p(s) + \lambda_j b(s)q(s)$ from the denominator of (6). Let $\alpha_j + \jmath\beta_j = \lambda_j M(\jmath\omega_0)$. Since $|T_{\min}(\jmath\omega_0)| > 1$, from Lemma 2 a) we know that $\alpha_j < -\frac{1}{2}$.
2) If $\alpha_j \leq -1$, then find γ_i such that $\gamma_i \geq \lambda_j$. Form $Z_{ij}(s)$ using such γ_i and λ_j. Then by c) in Lemma 2 for such $Z_{ij}(s)$ holds $|Z_{ij}(\jmath\omega_0)| \geq |Z_{ij}(0)|$.
3) If $-1 < \alpha_j \leq -\frac{1}{2}$, then find γ_i such that $\gamma_i \leq \lambda_j$. Form $Z_{ij}(s)$ using these γ_i and λ_j. Then by Lemma 2 d) $|Z_{ij}(\jmath\omega_0)| \geq |Z_{ij}(0)|$.
4) Form as much $Z_{ij}(s)$'s as possible using the steps 2) and 3). Use $(d_{CO}+1)$ remaining terms $a(s)p(s)+\lambda_j b(s)q(s)$ to form $T_j(s)$.

Lemma 1 f) allows us to find $(N - d_{CO} - 3)$ $Z_{ij}(s)$'s to satisfy either c) or d) in Lemma 2 — we pair γ_i with λ_{i+2} for $\alpha_j \leq -1$ and γ_i with λ_i for $-1 < \alpha_j \leq 0.5$ (see Fig. 5). These $Z_{ij}(s)$'s all have gain greater than one at ω_0. The remaining two $Z_{ij}(s)$'s might have gain less than one. Since λ_j and γ_i are bounded, there is a lower bound ξ such that $|Z_{ij}(\jmath\omega_0)| \geq \xi$ for these two.

The transfer function $T_{CO}(s)$ given in (6) is using such T_j's and Z_{ij}'s written as

$$T_{CO}(s) = w_{CO}\prod_{i=2, j\in\mathcal{J}}^{N-d_{CO}-1} Z_{ij}(s)\prod_{j=2,j\notin\mathcal{J}}^{N}\frac{1}{\lambda_j}\prod_{j=2,j\notin\mathcal{J}}^{N} T_j(s). \tag{16}$$

The set $\mathcal{J}$ is the set of λ_j used to form some of Z_{ij}'s. The terms $w_{CO}\prod_{j=2,j\notin\mathcal{J}}^{N}\frac{1}{\lambda_j}$ and steady-state gain of $Z_{ij}(0)$ do not affect the shape of the magnitude frequency response, only its value.

Since $\|T_{\min}(s)\|_\infty > 1$, it follows from a) in Lemma 2 that for all transfer functions $T_j(s)$ we have $|T_j(\jmath\omega_0)| > 1$. Due to the lower and upper bounds on eigenvalues, there is a minimum $\zeta > 1$ of modulus frequency response $|T_j(\jmath\omega_0)|$, attained for some λ_j with $\lambda_{\min} \leq \lambda_j \leq \lambda_{\max}$. Then we get the lower bound on the modulus of product of $T_j(s)$ in (16) as $\prod_{j=2,j\notin\mathcal{J}}^{N}|T_j(\jmath\omega_0)| \geq \zeta^{d_{CO}+1}$. Clearly, this part of (16) scales exponentially with d_{CO}.

All but two blocks $Z_{ij}(s)$ amplify at ω_0, so $\prod_{i=1}^{N-d_{CO}-1}|Z_{ij}(\jmath\omega_0)| \geq \xi^2$ (excluding the steady-state gain) and the norm of $T_{CO}(s)$ is from (16) $\|T_{CO}(s)\|_\infty \geq \xi^2\, T_{CO}(0)\,\zeta^{d_{CO}}$. □

Appendix D

Proof of Theorem 3. First we prove that if $\|T_{\max}(s)\|_\infty = 1$, then $\|T_{CO}(s)\|_\infty = |T_{CO}(0)|$. As in the proof of Theorem 2, we will form Z_{ij}'s and T_j's in a suitable way. Let $\alpha_j + \jmath\beta_j = \lambda_j M(\jmath\omega_0)$ at some frequency ω_0. Since $\|T_{\max}(s)\|_\infty = 1$, it follows from Lemma 2 b) that $|T_j(\jmath\omega_0)| \leq 1\,\forall\omega_0$, $\forall\lambda_j \leq \lambda_{\max}$ and $\alpha_j \geq -\frac{1}{2}$, $\forall\omega_0$.

Using Lemma 1 f) we can pair all γ_i with unique λ_j such that $\gamma_i \geq \lambda_j$ to form $Z_{ij}(s)$. Then e) in Lemma 2 implies that $|Z_{ij}(\jmath\omega_0)| \leq |Z_{ij}(0)|$ for all i, j. Since $\alpha_j \geq -\frac{1}{2}$ for all ω_0, we have that $\|Z_{ij}(s)\|_\infty = |Z_{ij}(0)|$ for all pairs $\gamma_i \geq \lambda_j$. All remaining terms $T_j(s)$ in (16) by Lemma 2b) satisfy $|T_j(\jmath\omega_0)| \leq 1$ for all ω_0. Hence, all transfer functions in the product (16) have their norm norm less than or equal to one and $\|T_{CO}(s)\|_\infty = |T_{CO}(0)|$.

Now let us go back to bidirectional string stability. Consider $O \geq C$ and let r_C be the input at the control node. Then the first transfer function in (12) can be written as

$$\begin{aligned}\frac{y_O(s)}{y_{O-1}(s)} &= \frac{r_C(s)T_{C,O}(s)}{r_C(s)T_{C,O-1}(s)} = \frac{T_{C,O}(s)}{T_{C,O-1}(s)} \\ &= \frac{b(s)q(s)\prod_{j=1}^{N-d_{CO}-1} a(s)p(s) + \gamma_{j,O}\, b(s)q(s)}{\prod_{j=1}^{N-d_{CO}} a(s)p(s) + \gamma_{j,O-1}\, b(s)q(s)}. \end{aligned} \tag{17}$$

Let $\hat{L}_{O-1}$ and $\hat{L}_O$ be the submatrices of L corresponding to the paths from C to $O-1$ and from C to O, respectively. Their eigenvalues are $\gamma_{j,O-1}$ and $\gamma_{j,O}$, respectively. Beacuse of the fact that $\hat{L}_O$ is a submatrix of $\hat{L}_{O-1}$, the eigenvalues of $\hat{L}_{O-1}$ and $\hat{L}_O$ must interlace in a sense of f) in Lemma 1. We can pair $\gamma_{j,O-1}$ and $\gamma_{j,O}$ by Lemma 1 f) such that $\gamma_{j,O-1} \leq \gamma_{j,O}$ and form $Z_{ij}(s)$ as above. Then, $\left\|\frac{a(s)p(s)+\gamma_{j,O}b(s)q(s)}{a(s)p(s)+\gamma_{j,O-1}b(s)q(s)}\right\|_\infty \leq 1\,\forall j$. Only one term in (17) with a form $\frac{b(s)q(s)}{a(s)p(s)+\gamma_{i,O-1}b(s)q(s)}$ remains. Its $\mathcal{H}_\infty$ norm is less than its steady-state gain by b) in Lemma 2. The steady-state gain of $\frac{y_O(s)}{y_{O-1}(s)}$ is one, since by Theorem 1 the steady-state gain is identical for all the vehicles behind the control node. Hence, $\|\frac{y_O(s)}{y_{O-1}(s)}\|_\infty \leq 1$ for $C \leq O$.

The other direction ($C \geq O$) has the ratio of outputs with the same structure as (17), the only difference is its steady-state gain. It follows from (9) that the steady-state gain is

$$\frac{T_{C,O-1}(0)}{T_{C,O}(0)} = \epsilon_{O-1}\frac{\left(1+\sum_{i=1}^{O-3}\prod_{j=1}^{i}\epsilon_{O-j-1}\right)}{\left(1+\sum_{i=1}^{O-2}\prod_{j=1}^{i}\epsilon_{O-j}\right)} < 1. \tag{18}$$

Since the norm $\|y_{O-1}(s)/y_O(s)\|_\infty$ is at most 1, bidirectional string stability was proved. □

References

[1] V. Milanés, S. Shladover, J. Spring, C. Nowakowski, H. Kawazoe, and M. Nakamura, "Cooperative adaptive cruise control in real traffic situations," *IEEE Transactions on Inteligent Transportation Systems*, vol. 15, no. 1, pp. 296–305, 2014.

[2] E. Coelingh and S. Solyom, "All aboard the robotic road train," *IEEE Spectrum*, vol. 49, no. 11, pp. 34–39, 2012.

[3] G. Naus, R. Vugts, J. Ploeg, R. van de Molengraft, and M. Steinbuch, "Cooperative adaptive cruise control, design and experiments," in *American Control Conference 2010*, no. 1, 2010, pp. 6145–6150.

[4] A. A. Peters, R. H. Middleton, and O. Mason, "Leader tracking in homogeneous vehicle platoons with broadcast delays," *Automatica*, vol. 50, no. 1, pp. 64–74, oct 2014.

[5] H. Hao and P. Barooah, "On achieving size-independent stability margin of vehicular lattice formations with distributed control," *IEEE Transactions on Automatic Control*, vol. 57, no. 10, pp. 2688–2694, 2012.

[6] P. Barooah, P. G. Mehta, and J. P. Hespanha, "Mistuning-Based Control Design to Improve Closed-Loop Stability Margin of Vehicular Platoons," *IEEE Transactions on Automatic Control*, vol. 54, no. 9, pp. 2100–2113, sep 2009.

[7] R. H. Middleton and J. H. Braslavsky, "String Instability in Classes of Linear Time Invariant Formation Control With Limited Communication Range," *IEEE Transactions on Automatic Control*, vol. 55, no. 7, pp. 1519–1530, 2010.

[8] P. Seiler, A. Pant, and K. Hedrick, "Disturbance propagation in vehicle strings," *IEEE Transactions on Automatic Control*, vol. 49, no. 10, pp. 1835–1841, 2004.

[9] H. Hao, H. Yin, and Z. Kan, "On the robustness of large 1-D network of double integrator agents," in *American Control Conference (ACC), 2012, Montreal, Canada*, 2012, pp. 6059–6064.

[10] C. E. Cantos and J. J. P. Veerman, "Transients in the Synchronization of Oscillator Arrays," *Arxiv preprint:1308.4919v4*, pp. 1–11, 2014.

[11] I. Herman, D. Martinec, and J. J. P. Veerman, "Transients of platoons with asymmetric and different Laplacians," *Systems & Control Letters*, vol. 91, pp. 28–35, may 2016.

[12] F. M. Tangerman, J. J. P. Veerman, and B. D. Stosic, "Asymmetric decentralized flocks," *IEEE Transactions on Automatic Control*, vol. 57, no. 11, pp. 2844–2853, 2012.

[13] I. Herman, D. Martinec, Z. Hurak, and M. Sebek, "Nonzero Bound on Fiedler Eigenvalue Causes Exponential Growth of H-Infinity Norm of Vehicular Platoon," *IEEE Transactions on Automatic Control*, vol. 60, no. 8, pp. 2248–2253, 2015.

[14] J. J. P. Veerman, B. D. Stošić, and A. Olvera, "Spatial instabilities and size limitations of flocks," *Networks and Heterogeneous Media*, vol. 2, no. 4, pp. 647–660, sep 2007.

[15] H. Hao and P. Barooah, "Stability and robustness of large platoons of vehicles with double-integrator models and nearest neighbor interaction," *International Journal of Robust and Nonlinear Control*, vol. 23, pp. 2097–2122, 2012.

[16] D. Martinec, I. Herman, Z. Hurák, and M. Šebek, "Wave-absorbing vehicular platoon controller," *European Journal of Control*, vol. 20, pp. 234–248, 2014.

[17] B. Bamieh, M. R. Jovanović, P. Mitra, and S. Patterson, "Coherence in Large-Scale Networks: Dimension-Dependent Limitations of Local Feedback," *IEEE Transactions on Automatic Control*, vol. 57, no. 9, pp. 2235–2249, 2012.

[18] P. Wieland, R. Sepulchre, and F. Allgöwer, "An internal model principle is necessary and sufficient for linear output synchronization," *Automatica*, vol. 47, no. 5, pp. 1068–1074, May 2011.

[19] S. Fallat and C. Johnson, *Totally Nonnegative Matrices*. Princeton University Press, 2011.

[20] I. Herman, D. Martinec, and M. Sebek, "Zeros of Transfer Functions in Network Control with Higher-Order Dynamics," in *19th IFAC World Congress*, Cape Town, South Africa, 2014, pp. 9177–9182.

[21] H. Zhang, F. L. Lewis, and A. Das, "Optimal Design for Synchronization of Cooperative Systems: State Feedback, Observer and Output Feedback," *IEEE Transactions on Automatic Control*, vol. 56, no. 8, pp. 1948–1952, aug 2011.

[22] J. A. Fax and R. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1465–1476, Sep. 2004.

[23] S. K. Yadlapalli, S. Darbha, and K. R. Rajagopal, "Information flow and its relation to the stability of the motion of vehicles in a rigid formation," *IEEE Transactions on Automatic Control*, vol. 51, no. 8, pp. 1315–1319, 2006.

[24] J. Ploeg, N. van de Wouw, and H. Nijmeijer, "Lp String Stability of Cascaded Systems: Application to Vehicle Platooning," *IEEE Transactions on Control Systems Technology*, vol. 22, no. 2, pp. 786–793, 2014.

[25] F. Lin, M. Fardad, and M. R. Jovanović, "Optimal control of vehicular formations with nearest neighbor interactions," *IEEE Transactions on Automatic Control*, vol. 57, no. 9, pp. 2203–2218, 2012.

[26] J. Eyre, D. Yanakiev, and I. Kanellakopoulos, "A Simplified Framework for String Stability Analysis of Automated Vehicles," *Vehicle System Dynamics*, 1998.

[27] S. Darbha, "On the synthesis of controllers for continuous time LTI systems that achieve a non-negative impulse response," *Automatica*, vol. 39, no. 1, pp. 159–165, 2003.

[28] R. Horn and C. Johnson, *Matrix analysis*. Cambridge: Cambridge University Press, 1990.