Transfer functions in consensus systems with higher-order dynamics and external inputs

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***Abstract*—This paper considers transfer functions in consensus systems where agents have identical SISO dynamics of arbitrary order. The interconnecting structure is a directed graph. The transfer functions for various inputs and outputs are presented in simple product forms with a similar structure of the numerator and the denominator. This structure combines the network properties and the agent model in an explicit way. The link between a higher-order and a single-integrator dynamics is shown and the polynomials of the transfer function in the single-integrator system are related to the graph properties. These properties also allow to generalize a result on the minimal dimension of the controllable subspace to the directed graphs.**

I. INTRODUCTION

Distributed control has become a very intensive field of research. Numerous results for control of highway platoons, robot formations or synchronization of oscillators were published. To the standard consensus problem an exogenous input can be added, which could capture for instance the effect of a measurement noise, input or output disturbances [1]–[3], reference values [4] or even an input of the intruder [5].

Much effort has been invested in understanding the behavior of single integrator systems, especially of the consensus. The results reveal the effect the network structure has on the overall behavior. For example, the convergence time is related to the second smallest eigenvalue of the Laplacian matrix [6]. The effect of the network structure on the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms was investigated in [7].

In many systems (e.g., highway platoons or oscillators), the agent models are more complicated higher-order systems. In this case stability is a crucial issue. It was shown in the paper [8] that the overall formation of identical agents is stable if and only if it is stable for all eigenvalues of a Laplacian matrix. One of the approaches for stabilization is based on changing the gains when the graph topology changes [9], [10]. Also the concept of passivity guarantees stability (see [11], [12]).

Nevertheless, stability is not sufficient for good transients. Phenomena not seen in the single integrator dynamics can appear with higher-order dynamics. Well known is a so called string stability, which concerns amplification of the disturbance in a vehicular formation. Some of the works in this field using the properties of the transfer functions are [4], [13], [14]. The effect of noise on the rigidity of the formation, known as coherence, was studied in the paper [3].

When the effects of inputs are considered, a controllability becomes an issue. The results on controllability inferred from graph structure are shown in [15]. The bounds on the dimension of the controllable subspace for undirected graphs are provided in [16]. The paper [5] shows relations of controllability grammian to effort which an intruder has to exert to steer the network system. The optimization with respect to performance and controllability is proposed in [17].

Dynamic behavior and frequency response of a linear system are given by the poles and zeros of the transfer function from the input to the output. Their location also determines the response to some reference signal. The structure of poles of the transfer functions was investigated in [18] or [6]. While the location of poles of a network systems is now well understood, less attention was paid to the location of zeros.

One of the first papers considering the location of zeros in the consensus based algorithms was [19], considering a single integrator model of one agent and a symmetric communication structure. The paper [20] extends the results of [19] to the directed graphs and also shows relations of the zeros to the Laplacian matrix. Transfer functions and their margins in cyclic formations are discussed in [21]. The paper [22] shows that even a formation with stable poles can have zeros in the right half-plane. If the goal is to externally control the network system, such zeros complicate the feedback design.

In this paper we study transfer functions in a network system where one agent with a known input acts as a controlling node and some other agent, output of which is of interest, serves as an observing node. All the agents are modelled by SISO systems and they are interconnected over directed graphs using relative output feedback. We generalize the results of [19] and [20] to higher-order dynamics and directed graphs. The key results are:

1) A product form of the transfer function (Theorem 3) showing a similar structure of the poles and zeros. Such a product form expresses the transfer function as a series connection of systems with identical structure. The transfer function with a general input and output consists of two parts: a network part and an open-loop part (Theorem 7).
2) A graph theoretical representation of the polynomials in single-integrator system (Lemma 2) and the relation of the zeros to the Laplacian (Theorem 8). If there is only one path between the controlling and the observing node, then the zeros are obtained from the Laplacian matrix.
3) The minimal dimension of the controllable subspace is related to the maximal distance from the controlling node (Theorem 11).

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Since we work with an arbitrary LTI model, the results here do not tell us much about particular transient properties. The results should rather serve as tools for analysis in performance assessment in particular graph types. For instance, the product form of the transfer function allowed us easier analysis of a scaling of the $\mathcal{H}_\infty$ norm in vehicular platoons [4].

This paper extends our preliminary results in [23]. We add graph theoretic representations of all polynomials and different types of inputs and outputs are considered. Also a result on the minimal dimension of the controllable subspace is added.

Notation: We denote matrices with capital letters and a particular element in a matrix A is denoted as a_{ij}. All vectors are column vectors and are denoted with lowercase letters, the ith element of a vector v is v_i. Scalars are denoted by Greek letters. I is an identity matrix and a canonical basis vector is $e_i = [0, \ldots, 1, \ldots, 0]^T$ with 1 on the ith position. The symbol s used in transfer functions denotes the Laplace variable. The polynomials are denoted by lowercase letters and g_i is the coefficient at s^i in the polynomial $g(s)$ (the argument s is usually used with polynomials).

II. GRAPH THEORY

The network system interconnection (sharing of information) can be viewed as a *directed graph*. The graph $\mathcal{G}$ has a vertex set $\mathcal{V}(\mathcal{G})$ and an arc set $\mathcal{E}(\mathcal{G})$. The arc $\epsilon(\nu_j, \nu_i)$ is oriented, which means that the ith agent receives its information from the jth agent. A directed path π_{ij} from i to j of length $l(\pi_{ij})$ is a sequence of vertices and arcs $\nu_1, \epsilon_1, \nu_2, \epsilon_2, \ldots \nu_{l+1}$, where each vertex and arc can be used only once. The length (number of arcs) of the shortest path between i and j is called the distance δ_{ij} of vertices. A cycle is a path with the first and last vertices identical.

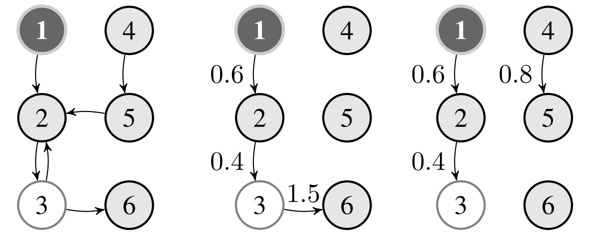
An adjacency matrix is defined as $A = [a_{ij}]$. Its entries a_{ij} are either zero if there is no arc from ν_j to ν_i or a positive number called weight if the arc is present. We also define the weight of the path as $\vartheta(\pi_{ij}) = \prod_{\epsilon(k,m)\in\pi_{ij}} a_{km}$. It is the product of weights of all arcs in the path. Similarly, we define the weight of a subset $\mathcal{G}'$ of a graph $\mathcal{G}$ as

$$\vartheta(\mathcal{G}') = \prod_{\epsilon(k,m)\in\mathcal{E}(\mathcal{G}')} a_{km}. \tag{1}$$

A directed tree is a subset of a graph without directed cycles. A diverging directed tree always has a path from one particular node called the root to each node in the tree. There is no directed path from the nodes in the diverging tree to the root and all the nodes except for the root have in-degree one. A forest $\bar{F}$ is a set of mutually disjoint trees. A spanning forest is a forest on all vertices of the graph (see [24] for an overview of directed trees). A diverging forest (out-forest) is a forest of diverging trees. Following the notation of [25] we denote $\mathcal{F}_k^{i\to j}$ the *set of all spanning diverging forests* with k arcs. Such a set must contain a tree with the root i which contains the node j. The weight of this set is

$$\vartheta(\mathcal{F}_k^{i\to j}) = \sum_{\bar{F}_k^{i\to j}\in\mathcal{F}_k^{i\to j}} \vartheta(\bar{F}_k^{i\to j}), \tag{2}$$

with the sum taken over all spanning forests $\bar{F}_k^{i\to j}$ in the set $\mathcal{F}_k^{i\to j}$. This is illustrated in Fig. 1.



(a) Orig. graph (b) First forest (c) Second forest

Fig. 1: Example of the set $\mathcal{F}_3^{1\to 3}$ of all spanning forests with three arcs with a tree diverging from the node 1 and containing 3. The weights of the two spanning forests are: (a) $\vartheta(\bar{F}_3^{1\to 3})_1 = 0.6 \cdot 0.4 \cdot 1.5 = 0.36$ and (b) $\vartheta(\bar{F}_3^{1\to 3})_2 = 0.6 \cdot 0.4 \cdot 0.8 = 0.192$. The weight of the set is $\vartheta(\mathcal{F}_3^{1\to 3}) = 0.192 + 0.36 = 0.552$.

Let Q_k be a matrix of spanning out-forests of $\mathcal{G}$ which have k arcs. The (i,j)th element $(q^k)_{ij}$ of Q_k is given as

$$(q^k)_{ij} = \vartheta(\mathcal{F}_k^{j\to i}). \tag{3}$$

It is the weight of the set of all spanning out-forests $\mathcal{F}_k^{j\to i}$ with k arcs containing i and diverging from the root j.

Let us denote $D = \operatorname{diag}\bigl(\deg(\nu_i)\bigr)$ the diagonal matrix of the sums of weights of the arcs incident to the vertex i. Then the Laplacian matrix $L \in \mathbb{R}^{N\times N}$ of a directed graph is defined as

$$L = D - A. \tag{4}$$

We denote the eigenvalues of the Laplacian as λ_i, $i = 1, \ldots, N$. All the eigenvalues have positive real part and there is always a zero eigenvalue of the Laplacian, i.e., $\lambda_1 = 0$ with the corresponding eigenvector $\mathbf{1}$ of all ones, i.e., $L\mathbf{1} = 0$.

In the paper we will use a version of Lemma 3.1 in [19]. Here we provide a different proof, as the original proof is valid only for commuting matrices and unweighted graphs.

Lemma 1. *For the elements of the powers of Laplacian holds*

$$(-L^m)_{ij} = \begin{cases} 0, & \text{for } m < \delta_{ji} \\ \vartheta(\mathcal{F}_k^{j\to i}), & \text{for } m = \delta_{ji} \end{cases}, \tag{5}$$

Proof. We will use the result [25, Proposition 8], which shows

$$(-L)^m = \sum_{k=0}^{m} \alpha_k Q_{m-k}, \tag{6}$$

with $\alpha_k \in \mathbb{R}$ being a constant. Since $(q^{m-k})_{ij}$ is the weight of $\mathcal{F}_{m-k}^{j\to i}$, the minimal number of arcs for any forest in the set to exist is the distance δ_{ji} from the node i to the node i. Hence, for $m < \delta_{ji}$, (i,j)th element of all Q_{m-k} is zero and therefore $(-L^m)_{ij}$ is also zero. For $m = \delta_{ji}$ the element $(-L^m)_{ij}$ is the sum of the weights of all shortest paths. ☐

III. SYSTEM MODEL

We consider a network system consisting of N identical agents which exchange information about their outputs (either using a communication or measurements). All are modelled as SISO systems, where dynamic controllers are used. Each agent is governed locally, therefore no central controller is used.

The plant model $G(s)$ (the model of an agent without the controller) is given as a transfer function of arbitrary order $G(s) = \frac{b(s)}{a(s)}$. The output of the ith plant is denoted as y_i. The plant model is driven by the output of the dynamic controller $R(s)$. The controller is generally given as a transfer function $R(s) = \frac{q(s)}{p(s)}$. The input to the controller is u_i. As the plant and the controller are connected in series, the agent model is described by the *scalar* open-loop transfer function

$$M(s) = G(s)R(s) = \frac{b(s)q(s)}{a(s)p(s)}. \tag{7}$$

The relative degree (the difference between the degree μ of the denominator and the degree η of the numerator) of $M(s)$ is denoted as $\chi = \mu - \eta$.

The neighbor of an agent i is defined as an agent j from which the agent i can obtain information about its output, that is, there exists an arc $\epsilon(\nu_j, \nu_i)$ in the graph $\mathcal{G}$. The relative error of the ith agent is defined as $\tilde{e}_i = \sum_{j\in\mathcal{N}(i)}(y_j - y_i)$, where $\mathcal{N}(i)$ denotes the set of neighbors of the ith agent.

Apart from the relative error $\tilde{e}_i$, an exogenous input r_i can be acting at the input of the controller. The total input to the controller thus is

$$u_i = \tilde{e}_i + r_i = \Big(\sum_{j\in\mathcal{N}(i)} (y_j - y_i)\Big) + r_i, \tag{8}$$

The input r_i can be, for instance, the sum of reference values or some other external signal such as error in measurement, disturbance etc. We treat r_i as a general signal.

A. Problem statement

The stacked vector of all inputs to the open loops is

$$u(s) = -Ly(s) + r(s), \tag{9}$$

with $u = [u_1, \ldots, u_N]^T$, $y = [y_1, \ldots, y_N]^T$ and $r = [r_1, \ldots, r_N]^T$. The matrix L is the graph Laplacian in (4).

Now we can write the model of the overall formation as

$$y(s) = M(s)u(s) = M(s)\left[-Ly(s) + r(s)\right]. \tag{10}$$

We are interested in how an exogenous input acting at one selected agent affects the output of another agent. We assume that there is only one input r_c, acting at the input of the agent with index c. That is, the input vector equals $r = [0, \ldots, 0, r_c, 0, \ldots, 0]^T = e_c r_c$. We will call the agent with index c a *controlling agent*.

The output of interest is the output y_o of the agent with index o, i. e. the output vector is $y = [0, \ldots, 0, y_o, 0, \ldots, 0]^T = e_o y_o$. We call the agent with index o an *observing agent*. The indices c and o can be arbitrary. We will use the statement “from c to o” with the meaning of “from the input r_c acting at the agent c to the output y_o of the agent o”.

Define a transfer function $T_{co}(s)$ as

$$T_{co}(s) = \frac{y_o(s)}{r_c(s)}. \tag{11}$$

Problem statement. Consider the transfer function $T_{co}(s)$ for a network of SISO agents connected by a directed graph. We study the structure of $T_{co}(s)$ and analyze how does $T_{co}(s)$ depend on the open loop model $M(s)$, the choice of agents c and o and the interconnection Laplacian L.

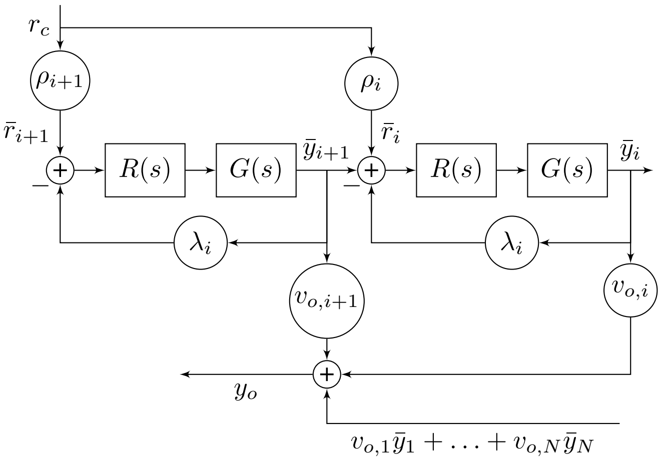


Fig. 2: One diagonal block for the case of Jordan block of size 2. The eigenvalue λ_i acts as a gain in the feedback. Only one closed loop is present if the Jordan block has a size one.

B. Block diagonalization

We can block diagonalize the system (10) using the transformation $y = V\bar{y}$. The matrix $V = [v_{ij}]$ is a matrix of (generalized) eigenvectors of the Laplacian, i. e. $LV = \Lambda V$ with Λ being the Jordan form of L. With such a transform, the model has a form

$$V\bar{y}(s) = M(s)\left[-LV\bar{y}(s) + r(s)\right]. \tag{12}$$

Separating $\bar{y}$ on the left-hand side using $\Lambda = V^{-1}LV$ yields

$$\left[I + \Lambda M(s)\right]\bar{y}(s) = M(s)V^{-1}r(s). \tag{13}$$

We can define the transformed input to the system $\bar{r}(s) = V^{-1}r(s)$. Since $M(s)$ is a scalar transfer function, (13) is a block diagonal system, where each block has a size of a Jordan block corresponding to an eigenvalue λ_i of L. If the Jordan block for the eigenvalue λ_i has a size 1, then it can be written using a transfer function

$$T_i(s) = \frac{\bar{y}_i(s)}{\bar{r}_i(s)} = \frac{M(s)}{1 + \lambda_i M(s)} = \frac{b(s)q(s)}{a(s)p(s) + \lambda_i b(s)q(s)} \tag{14}$$

$T_i(s)$ is an output feedback system with a feedback gain λ_i. If, on the other hand, the block in (13) corresponds to a Jordan block of size 2, then its output can be written as the output of a series connection of identical blocks, such as

$$\bar{y}_i(s) = \frac{M(s)}{1 + \lambda_i M(s)}\left(\bar{r}_i(s) + \frac{M(s)}{1 + \lambda_i M(s)}\bar{r}_{i+1}(s)\right). \tag{15}$$

This easily generalizes to larger Jordan blocks. The structure is shown in Fig. 2.

For simplicity, the derivations throughout the paper will be shown only for the case where all Jordan block in Λ are simple — the eigenvalues λ_i have the same algebraic and geometric multiplicity. All the proofs can be conducted the same way for blocks of larger size and all the results remain valid.

If the eigenvalue λ_i is simple, the input to the ith diagonal block in (13) is the ith element of $\bar{r}$ and equals $\bar{r}_i = e_i^T V^{-1} e_c r_c = \rho_i r_c$ with $\rho_i = e_i^T V^{-1} e_c = (V^{-1})_{ic}$.

Thus, the input r_c enters the block $T_i(s)$ through the gain ρ_i and from (14) $\bar{y}_i(s) = T_i(s)\rho_i r_c(s)$. The output of the ith agent can be obtained using the outputs of the blocks as $y_i(s) = \sum_{j=1}^{N} v_{ij}\bar{y}_j(s)$. By setting $\bar{y}_j(s) = T_i(s)\rho_i r_c(s)$ in the previous equation, the output of the observing node is

$$y_o(s) = \left[\sum_{i=1}^{N} v_{oi}\rho_i T_i(s)\right] r_c(s) = T_{co}(s) r_c(s). \tag{16}$$

This also expresses the transfer function $T_{co}(s)$ in (11).

IV. TRANSFER FUNCTIONS IN GRAPHS

In this section we derive the structure of the transfer function $T_{co}(s)$ between the input r_c of the controlling node and output of the observing node y_o.

A. Single integrator dynamics

Before investigating the general case with higher-order dynamics, let us discuss a standard single-integrator case. We will later in the paper relate it to the higher-order dynamics. For the single single-integrator case $M(s) = \frac{1}{s}$ and the state-space description of the network system is $\dot{x} = -Lx + e_c r_c, \quad y_o = e_o^T x$. Let the single-integrator transfer function from r_c to y_o be a fraction of two polynomials as

$$T_{co}(s) = \frac{h(s)}{g(s)}. \tag{17}$$

The denominator polynomial $g(s)$ is given as

$$g(s) = \det(sI_N + L) = s^N + g_{N-1}s^{N-1} + \ldots + g_1 s + g_0. \tag{18}$$

$g(s)$ is a characteristic polynomial of $-L$. The roots of g (i. e., the poles of $T_{co}(s)$ for single integrator dynamics) are $-\lambda_i$, the eigenvalues of $-L$. The coefficient $g_0 = 0$ because there is always a zero eigenvalue of $-L$. If the zero eigenvalue is simple, it is known that the coefficients are

$$g_{N-1} = \sum_{i=1}^{N}\lambda_i, \;\; g_{N-2} = \sum_{i=1, j=1, i\neq j}^{N}\lambda_i\lambda_j, \;\; \ldots, \quad g_1 = \prod_{i=2}^{N}\lambda_i. \tag{19}$$

The other terms g_k are sums of all products of k eigenvalues.

The numerator polynomial is given as $h(s) = h_{N_n}s^{N_n} + \ldots + h_1 s + h_0$. It was shown in [19], [20] that $N_n = N - \delta_{co} - 1$. We denote the N_n roots of $h(s)$ as $-\gamma_i$, so

$$h(s) = h_{N_n}(s+\gamma_1)(s+\gamma_2)\ldots(s+\gamma_{N_n}). \tag{20}$$

The coefficients of g and h have a graph-theoretic representation. For the denominator polynomial $g(s)$ they are given by [25, Proposition 2] as $g_i = \vartheta\left(\mathcal{F}_{N-i}\right)$, which is the weight of the set of all diverging forests in the graph with $N - i$ arcs. This also explains why $g_0 = 0$ — there is no spanning forest with N arcs (there has to be a cycle in N arcs).

The numerator polynomial can be calculated as

$$h(s) = e_o^T \operatorname{adj}(sI + L)\, e_c, \tag{21}$$

which is the o, cth cofactor of $(sI + L)$. It is shown in [25, Proposition 3] that

$$\operatorname{adj}(sI + L) = \sum_{i=0}^{N} Q_i s^{N-i-1}, \tag{22}$$

Lemma 2. *The coefficients h_i are given as $h_i = \vartheta(\mathcal{F}^{c\to o}_{N-i-1})$.*

Proof. The polynomial $h(s)$ equals the o, c element of $\operatorname{adj}(sI + L)$ (21). The coefficient at s^i in $h(s)$ is by (22) equal to the o, c element of matrix Q_{N-i-1}, i.e., $h_i = q_{oc}^{N-i-1}$. By (3) this element also must be equal to $\vartheta(\mathcal{F}^{c\to o}_{N-i-1})$. ☐

This indicates that the coefficients h_i are given as the weights of the set of all spanning diverging forests with $N - i - 1$ arcs which contain o and diverge from c. In the case of unweighted graph the weight reduces to the number of such out-forests.

While the coefficients in the denominator polynomial correspond to all diverging forests with the given number of arcs, the numerator polynomial takes only those spanning out-forests containing the controlling and the observing nodes.

B. Higher order dynamics

Now let us go back to higher-order systems. We have the definition of $-\gamma_i$ as the roots of $h(s)$ in (20), so we can state the main theorem of the paper. It relates the single-integrator systems to the higher-order dynamics.

Theorem 3. *The transfer function $T_{co}(s)$ can be written as*

$$T_{co}(s) = \vartheta_{co}\frac{[b(s)q(s)]^{1+\delta_{co}}\displaystyle\prod_{i=1}^{N-1-\delta_{co}}\big(a(s)p(s) + \gamma_i b(s)q(s)\big)}{\displaystyle\prod_{i=1}^{N}\Big(a(s)p(s) + \lambda_i b(s)q(s)\Big)}, \tag{23}$$

where $\vartheta_{co} = h_{N-\delta_{co}-1}$ is the sum of weights of all shortest paths from c to o, δ_{co} is the distance from c to o and the gains $-\gamma_i$ defined in (20) is the root of $h(s)$.

The proof can be found in the appendix. It is clear that the roots $-\gamma_i$ of the single-integrator numerator polynomial $h(s)$ have the same role as the roots $-\lambda_i$ of the denominator polynomial $g(s)$. As can be seen, the structure of the terms in the numerator and the denominator of (23) is $a(s)p(s) + k\; b(s)q(s)$, where $k = \lambda_i$ in the denominator and $k = \gamma_i$ in the numerator. In addition, such structure is the same as the structure of the characteristic polynomial of an output-feedback system with the open loop $M(s) = k\,\frac{b(s)q(s)}{a(s)p(s)}$ with the gain $k = \lambda_i$ or $k = \gamma_i$.

If both γ_i and λ_i are real, the poles and zeros of (23) lie on the root-locus curve (see Fig. 7 for an example). The root-locus curve is defined as a location of roots of $a(s)p(s) + k\, b(s)q(s)$ as a function of $k \in (0, \infty)$. Note that both the terms in the numerator and denominator of (23) have this form.

A particular case of the product form (23) was shown in [1, Proposition 3], where the authors considered single integrators ($M(s) = 1/s$) and unidirectional interaction.

The product form in (23) can be written also as

$$T_{co}(s) = \vartheta_{co}\prod_{i=1}^{N-\delta_{co}-1} Z_i(s)\prod_{j=N-\delta_{co}}^{N} T_j(s), \tag{24}$$

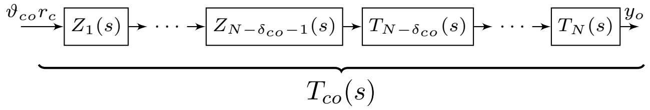


Fig. 3: Series form of the transfer function $T_{co}(s)$.

with

$$Z_i(s) = \frac{a(s)p(s) + \gamma_i b(s)q(s)}{a(s)p(s) + \lambda_i b(s)q(s)}, \tag{25}$$

$$T_j(s) = \frac{b(s)q(s)}{a(s)p(s) + \lambda_j b(s)q(s)}. \tag{26}$$

All eigenvalues λ_i must be used, so $N - \delta_{co} - 1$ of them go to $Z_i(s)$ and the remaining $\delta_{co} + 1$ to $T_j(s)$. The transfer functions $Z_i(s)$ are biproper and the numerator differs from the denominator only in the multiplication factor γ_i. The transfer functions $T_j(s)$ are standard output feedback systems in (14).

The network system (10) of identical agents with arbitrary interconnection was transformed in equation (24) to a series connection (product of transfer functions) of non-identical (but structured) subsystems. In many cases, such as in determining a frequency response, the series connection is much easier to analyze [4]. The series connection is illustrated in Fig. 3.

As the numerator of the open loop $b(s)q(s)$ is present for $\delta_{co} + 1$ times in (23), we have the following corollary.

Corollary 4. *The transfer function $T_{co}(s)$ has $\delta_{co}+1$ multiple zeros at the locations of the zeros of the open loop, i. e. roots of $b(s)q(s) = 0$.*

These zeros can be partly chosen by the designer of the network, since he can choose the controller numerator $q(s)$ freely. Contrary, the zeros of $Z_i(s)$ are given by the interconnection matrix in the same way as the poles are.

A relative degree comes immediately from Theorem 3.

Corollary 5. *Let χ be the relative degree of $M(s)$. Then the relative degree χ_{co} of $T_{co}(s)$ is $\chi_{co} = (\delta_{co} + 1)\chi$.*

Proof. There is $N - \delta_{co} - 1$ blocks of type $Z_i(s)$ in (23), which have relative degree 0. Then there is $\delta_{co}+1$ terms $T_i(s)$ which have relative degree χ. Hence, $\chi_{co} = (\delta_{co} + 1)\chi$. □

The relative degree strongly affects the transients. The transfer functions $Z_i(s)$ have relative order 0, so the input gets directly to the output. The $\delta_{co} + 1$ terms $T_j(s)$ slow down the transient. Quite clearly, the further the control and observer nodes are from each other, the slower the transient will be.

Another immediate result is the steady-state value.

Corollary 6. *For at least one integrator in the open loop, the steady-state gain of any transfer function in the formation is*

$$T_{co}(0) = \vartheta_{co} \frac{\prod_{i=1}^{N-1-\delta_{co}} \gamma_i}{\prod_{i=1}^{N} \lambda_i}. \tag{27}$$

Proof. For at least one integrator in the open loop, $a(0)p(0) = 0$. After plugging this to (23), the result follows. □

At least one integrator in the open loop is a common requirement to allow an uncontrolled network system to have a nonzero equilibrium.

The most important fact following from the Corollary 6 is that the steady-state gain *does not depend* on the open-loop model, as long as there is at least one integrator in $M(s)$. To change the steady-state value, the interconnection structure must be modified.

We will discuss two cases. First, assume that $\gamma_i \neq 0,\ \forall i$. Then the eigenvalue $\lambda_1 = 0$ of the Laplacian in the denominator makes the steady-state gain infinite. This happens when there is no independent leader in the network system.

If, on the other hand, there is $\gamma_1 = 0$, the eigenvalue at the origin $\lambda_1 = 0$ will be cancelled. As a result, the steady-state value is bounded. The presence of $\gamma_1 = 0$ is usually caused by the presence of an independent leader in the system. Such a leader cannot be controlled from the network system, hence the zero eigenvalue will be uncontrollable, causing the pole-zero cancellation.

V. GENERAL TRANSFER FUNCTIONS

So far we have analyzed properties of a transfer function from the input of the controller of agent c to the output of the agent o. However, we might also be interested in a transfer function from a general input w_c at the controlling node to a general output z_o of the observing node. In this section we show that the general transfer function has two parts: an open-loop part and a network part.

There is always at least one zero eigenvalue of L, therefore in (23) $a(s)p(s) + \lambda_1 b(s)q(s) = a(s)p(s)$, which is the denominator of the open loop $M(s)$. Also at least one numerator polynomial of the open loop $b(s)q(s)$ is present in $T_{co}(s)$. Then the transfer function in (23) can be written as

$$\begin{aligned} T_{co}(s) &= \vartheta_{co} M(s) \frac{\big(b(s)q(s)\big)^{\delta_{co}} \displaystyle\prod_{j=1}^{N_n} a(s)p(s) + \gamma_j b(s)q(s)}{\prod_{i=2}^{N} a(s)p(s) + \lambda_i b(s)q(s)} \\ &= M(s) S_{co}(s), \end{aligned} \tag{28}$$

where

$$S_{co}(s) = \vartheta_{co} \frac{\big(b(s)q(s)\big)^{\delta_{co}} \prod_{j=1}^{N_n} \big(a(s)p(s) + \gamma_j b(s)q(s)\big)}{\prod_{i=2}^{N} \big(a(s)p(s) + \lambda_i b(s)q(s)\big)} \tag{29}$$

is the network part of $T_{co}(s)$ and $M(s)$ is the open-loop.

Let $M_s(s)$ be the transfer function in open-loop of one agent from the desired input w_i (e. g. a reference or a disturbance) to the desired output z_i of same agent, i. e., $M_s(s) = z_i(s)/w_i(s)$.

Theorem 7. *The transfer function $T_{wz,co}(s)$ from the input of the controlling agent w_c to the output z_o of the observing agent is given as*

$$T_{wz,co}(s) = \frac{z_o(s)}{w_c(s)} = M_s(s) S_{co}(s). \tag{30}$$

Proof. Consider first that the controlling and the observing nodes are collocated ($c = o$). Then by changing the input

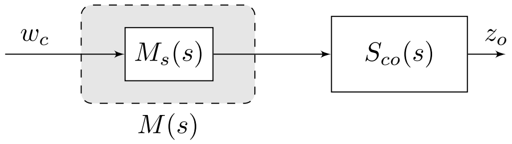


Fig. 4: Two parts of transfer functions between c and o for general input and output.

from r_c to w_c and the output from y_o to z_o we just change the direct branch of the transfer function $T_{cc}(s)$. The direct branch is then $M_s(s)$ instead of $M(s)$. The network (feedback) part S_{cc} in (29) remains unchanged. That is,

$$\frac{z_c(s)}{w_c(s)} = M_s(s)S_{cc}(s). \tag{31}$$

Consider now that c and o are not collocated. Define two transfer functions of a single agent:

$$M_1(s) = \frac{y_i(s)}{w_i(s)}, \quad M_2(s) = \frac{z_i(s)}{r_i(s)}. \tag{32}$$

Note that $\frac{M_1(s)M_2(s)}{M(s)} = \frac{[y_i(s)/w_i(s)]\,[z_i(s)/r_i(s)]}{y_i(s)/r_i(s)} = M_s(s)$.

The transfer function from $y_c(s)$ to $y_o(s)$ using the input r_c is

$$\frac{y_o(s)}{y_c(s)} = \frac{r_c(s)M(s)S_{co}(s)}{r_c(s)M(s)S_{cc}(s)} = \frac{S_{co}(s)}{S_{cc}(s)}. \tag{33}$$

From (31) we get $y_c(s) = M_1(s)S_{cc}w_c(s)$. Plugging this to (33) gives

$$y_o(s) = \frac{S_{co}(s)}{S_{cc}(s)}M_1(s)S_{cc}(s)w_c(s) = M_1(s)S_{co}(s)w_c(s). \tag{34}$$

Similarly, the transfer function from $z_o(s)$ to $y_o(s)$ is

$$\frac{y_o(s)}{z_o(s)} = \frac{r_o(s)M(s)S_{oo}(s)}{r_o(s)M_2(s)S_{oo}(s)} = \frac{M(s)}{M_2(s)}, \tag{35}$$

therefore $y_o(s) = M(s)/M_2(s)z_o(s)$. Plugging this to (34) and separating $z_o(s)$ yields

$$z_o(s) = \frac{M_1(s)M_2(s)}{M(s)}S_{co}(s)w_c(s) = M_s(s)S_{co}(s)w_c(s). \tag{36}$$

The transfer function $T_{\mathrm{wz},co}(s)$ follows. ☐

The general structure is shown in Fig. 4. It follows that each transfer function in the network system is given by two parts:

1) the network part $S_{co}(s)$, which is the same for all transfer functions with the same c and o nodes and is given by the interconnection,
2) the open loop part $M_s(s)$, which depends on the inputs and outputs of interest.

A. Disturbances

First we analyze an input disturbance $d_{\mathrm{in}c}$, acting at the input of the plant. The modified open-loop transfer function is $M_s(s) = G(s)$. Then the transfer function is

$$T_{\mathrm{in},co}(s) = \frac{y_o(s)}{d_{\mathrm{in}c}(s)} = G(s)S_{co}(s). \tag{37}$$

It is clear that $T_{co}(s)$ and $T_{\mathrm{in},co}(s)$ differ only in the presence of transfer function of the controller and $T_{co}(s) = R(s)T_{\mathrm{in},co}(s)$.

The output disturbance d_{out} changes the output of the plant of the jth agent as $y_j = \bar{y}_j + d_{\mathrm{out}j}$, where $\bar{y}_i$ is the output of the agent without disturbance. In this case $M_s(s) = 1$, so the transfer function for output disturbance is

$$T_{\mathrm{out},co}(s) = \frac{y_o(s)}{d_{\mathrm{out}c}(s)} = S_{co}(s). \tag{38}$$

VI. RELATIONS TO SINGLE-INTEGRATOR CASE

In this section we provide some results for the single-integrator case. They easily generalize to higher-order dynamics, because of the fact that γ_i, the gain in the closed loop in (23), is the same as the zero in the single-integrator dynamics. Let us denote $\bar{L}^k_{i:j}$ as a matrix which is obtained from L by deleting the rows and columns corresponding to the vertices on the kth path from vertex i to j.

The simplest case is when the controller node and observer nodes are collocated, i.e. $c = o$. Then, as shown in [19], [22], [23], the zeros are given as eigenvalues of $\bar{L}^1_{(c:c)}$ and the numerator polynomial is

$$h(s) = \det(sI + \bar{L}^1_{(c:c)}). \tag{39}$$

The spectrum of this reduced Laplacian (also known as a grounded Laplacian) is discussed in [26].

The next theorem was independently discovered in [20] using purely algebraic techniques. Here we provide a graph-theoretic proof.

Theorem 8. *If there is only one path between the controlling node and the observing node, then*

$$h(s) = \vartheta_{co} \det\left(sI + \bar{L}^1_{c:o}\right). \tag{40}$$

The roots $-\gamma_i$ of $h(s)$ are the eigenvalues of $-\bar{L}^1_{c:o}$.

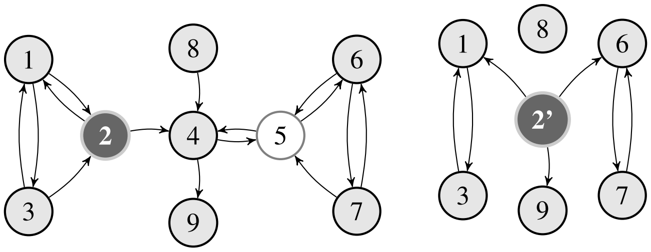
Proof. Recall that by (21) $h(s)$ equals (o, c) cofactor of $sI+L$. By Lemma 2 coefficients h_i of $h(s)$ are the weights of the set of all spanning diverging forests with the root c and containing o having $N - i - 1$ arcs, therefore $h_i = 0$ for $i \geq N - \delta_{co}$. In addition, the path from c to o must be present in every spanning forest with more than δ_{co} arcs.

The proof will be shown in several steps of modifying the original graph $\mathcal{G}$ and constructing a new one $\mathcal{G}'$ with the preserved polynomial $h(s)$.

1) Remove all the arcs converging to the path π_{co} from c to o. They cannot be part of any forest diverging from c and containing o.
2) Since by the assumption there is only one path between c and o, the path π_{co} is present in each forest in Lemma 2 and the weight $\vartheta_{co} = \vartheta(\pi_{co})$ of the path must be present in all coefficients h_i. We can write

$$\begin{aligned} h(s) = \vartheta_{co}\bar{h}(s) = \vartheta_{co}\big(&s^{N-1-\delta_{co}} \\ &+ \mu_{N-2-\delta_{co}}s^{N-2-\delta_{co}} + \ldots + \mu_0\big). \end{aligned} \tag{41}$$

This factoring acts as removing the arcs on the path from the graph.



(a) Before reduction

(b) After reduction

Fig. 5: Graph reduction without changing $h(s)$. The controlling node is the node 2, observing is 5.

3) Now we want to find a matrix of which $\bar{h}(s)$ is a characteristic polynomial. By factoring the weight of the path, we identified (created one from many) the vertices on the path into only one new vertex c'. All arcs connected to the path are now connected to the new vertex c'. The controlling and observing nodes were collocated. Denote such a new graph as $\mathcal{G}'$ with the number of vertices $N(\mathcal{G}') = N - \delta_{co}$. The process of such graph reduction is illustrated in Fig. 5.
4) The coefficients μ_i in (41) are the weights of the set of all spanning forests in the reduced graph $\mathcal{G}'$, diverging from c' with $N(\mathcal{G}') - i - 1$ arcs. Then, by (21-22), the polynomial $\bar{h}(s)$ equals the (c', c') cofactor of $(sI_{N-\delta_{co}} + \bar{L}^1_{c:o-1})$. Since the observing and controlling nodes are collocated in the modified graph $\mathcal{G}'$, we can use (39) to remove also the node c' from the graph.

In step 3 we deleted all nodes on the path except for the node c. In the last step we were also able to eliminate the controlling node, so the polynomial $h(s)$ can be calculated as

$$h(s) = \vartheta_{co} \det(sI + \bar{L}^1_{c:o}). \tag{42}$$

☐

The theorem allows to find γ_i directly from the submatrix of the Laplacian. The real part of γ_i is positive, since the matrix $\bar{L}^1_{c:o}$ is still an M-Matrix [27]. In addition, if L is a symmetric matrix and the conditions in Theorem 8 hold, then γ_i interlace with λ_i due to the Cauchy interlacing theorem [28].

The second theorem is an extension of the previous one.

Theorem 9. *Let $p(\mathcal{G})_{c,o}$ be the number of paths from the node c to the node o. Then the numerator polynomial $h(s)$ in (17) is given as a sum of characteristic polynomials of $\bar{L}^i_{c:o}$ corresponding to the individual paths π^i_{co}, i. e.*

$$h(s) = \sum_{i=1}^{p(\mathcal{G})_{c,o}} \vartheta(\pi^i_{co}) \det(sI + \bar{L}^i_{c:o}), \tag{43}$$

Proof. Since there are $p(\mathcal{G})_{c,o}$ paths between the nodes, there are also $p(\mathcal{G})_{c,o}$ basic trees diverging from c and containing o (they can have different lengths). For each of the paths Theorem 8 must hold. Let us denote the weight of spanning forests with $N - \delta^k_{co} - 1 - i$ arcs corresponding to the path k with length δ^k_{co} as h^k_i. Since the paths are distinct, also the spanning forests corresponding to the paths will be distinct and the total weight of the set $\mathcal{F}^{i \to j}_k$ is the sum of the weights of the individual trees. Then each coefficient in $h(s)$ is a sum of the weights of the trees corresponding to each path, i. e.

$$h_i = \sum_{k=1}^{p(\mathcal{G})_{c,o}} h^k_i. \tag{44}$$

Equation (43) then follows from (44) using Theorem 8. ☐

TABLE I: Controllable subspaces for some typical undirected graphs with N vertices.

Graph	c node	$\max_i d_{ci}$	Dim. of ctrb. subs.
Star graph	central	1	2
Path graph	end node	$N-1$	N
Path graph	central node	$N/2$	$N/2+1$

A. Multiple controlling nodes

Instead of one controlling node c we can have a set $\mathcal{S}_c = \{c_1, c_2, \ldots, c_{N_c}\}$ of N_c controlling nodes to which the same signal is fed (for instance, the leader connected to more agents). Then the numerator polynomial is simply given as a sum of polynomials for individual controlling nodes.

Lemma 10. *The polynomial $h(s)$ for the set of controlling nodes $\mathcal{S}_c$ is equal to $h(s) = \sum_{i=1}^{N_c} h_i(s)$, where $h_i(s)$ is the polynomial when the input is fed only to the ith agent.*

Proof. The proof can be obtained using the same arguments of mutually exclusive forests as in the proof for Theorem 9. ☐

Suppose that $c_n \in \mathcal{S}_c$ is the node in $\mathcal{S}_c$ with the shortest distance to the observing node. Then the relative degree of the transfer function $T_{co}(s)$ between $\mathcal{S}_c$ and o with agents having higher order-dynamics is $\chi_{co} = (\delta_{c_n o} + 1)\chi$. This follows since the degree of the sum of polynomials is the degree of the polynomial of the highest degree.

B. Minimal dimension of a controllable subspace

From equation (23) it follows that if the single-integrator case is uncontrollable, so are all the systems with higher order dynamics (we use an output feedback). The following result is an extension of [16, Thm. 2] to directed graphs.

Theorem 11. *Let $\max_i d_{ci}$ be the maximal distance to some of the other nodes from the controlling node c. Then for the dimension of the controllable subspace $\operatorname{rank}(\mathcal{C})$ of single integrator dynamics holds $\operatorname{rank}(\mathcal{C}) \geq \max_i d_{ci} + 1$.*

Proof. Let us denote the furthest node from c as f and the distance of f from c as $d_f = \max_i d_{ci}$. Let L^i_c denote the cth column in L^i. Let the vertices on the shortest path from c to f be labeled as $\nu_0, \ldots, \nu_{d_f}$ and the distance of ν_i from c as δ_i. By Lemma 1 the ν_ith element in L^j_c is zero for all $j < d_i$ and is nonzero for $j \geq d_i$. Therefore, $L^{d_i}_c$ is linearly independent of L^j_c for $j < d_i$ and $d_i = 0, \ldots, d_f$. Consequently, all columns $[L^0_c, L^1_c, \ldots, L^{d_f}_c]$ must be linearly independent.

The controllability criterion matrix is defined as $\mathcal{C} = [L^0_c, L^1_c, \ldots, L^N_c]$. By previous development we know that at least $L^0_c, \ldots L^{d_f}_c$ are linearly independent, hence $\operatorname{rank}(\mathcal{C}) \geq d_f + 1$. ☐

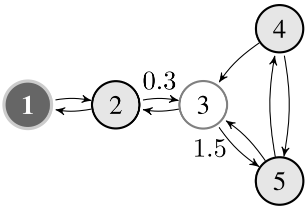


Fig. 6: Directed graph used in the example.

Of course, the controllable subspace can be much greater than indicated by this theorem and our result can be very conservative. The bound is achieved for some graphs and controlling nodes, as shown in Table I. Some further discussion of the tightness of the bound is in [16, Remark 2]. Theorem 11 gives a strong structural controllability, since it does not depend on the weights of the arcs. By any choice of the nonzero weights of arcs, the controllable subspace cannot have smaller dimension than $\max_i d_{ci}+1$. Structural controllability is described, e.g., in [17], [29].

Surprisingly, the more distant node exists in a graph, the greater the guaranteed dimension of the controllable subspace. On the other hand, it was shown in [30] that the transient time grows with the maximal distance from the control node. Similarly, at least for a path graph it follows from [2] that the external input should be applied to the agent where it minimizes the maximal distance. This is also confirmed by the relative degree in Corollary 5 — the higher the degree, the slower is the information spread. However, in this case the node has the smallest guaranteed controllable subspace. An optimization procedure for the tradeoff between performance and controllability is presented in [17].

VII. ILLUSTRATIVE EXAMPLE

Consider a directed and weighted graph with five nodes shown in Fig. 6 (The arcs without a weight shown have a weight one). The plant is $G(s) = 1/s$, the controller is $R(s) = (s+1)/s$ (a PI controller applied to a single integrator). The open-loop model is $M(s) = \frac{s+1}{s^2}$. Let us choose the controlling node $c = 1$ and the observing node $o = 3$. The transfer function is

$$T_{13}(s) = 0.3\frac{(s+1)^3\prod_{i=1}^{2}(s^2+\gamma_i s+\gamma_i)}{\prod_{i=1}^{5}(s^2+\lambda_i s+\lambda_i)}. \tag{45}$$

with $\lambda = \{0, 0.39, 2, 2.72, 3.69\}$ and $\gamma = \{0.5, 3\}$. As indicated by (23), the terms in the numerator and the denominator products have the structure of $a(s)p(s)+kb(s)q(s)$. Moreover, since the distance between the nodes 1 and 3 is 2, there is also $(s+1)^{2+1}$ in the numerator, as follows from Corollary 4. The weight of the path from the node 1 to 3 is 0.3 (the product of the weights of the arcs). The gains λ_i can be obtained as the eigenvalues of the Laplacian matrix

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -0.3 & 2.3 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1.5 & -1 & 2.5 \end{bmatrix}. \tag{46}$$

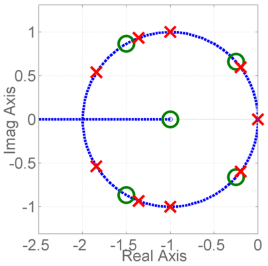


Fig. 7: Poles (crosses) and zeros (circles) of $T_{13}(s)$ in the graph in Fig. 6. The root-locus curve for $M = (s+1)/s^2$ is dashed.

The gains γ_i in the numerator can be obtained as the negatives of the roots of the polynomial $h(s) = s^2+3.5s+1.5$. Since there is only one path between c and o, we can use Theorem 8 to calculate the polynomial $h(s)$. It equals the characteristic polynomial of a matrix $\bar{L}^1_{(1:3)}$, obtained from L by deleting the rows and columns with indices $1, 2, 3$ of the vertices on the path from 1 to 3. The polynomial is given as

$$h(s) = \det\left(sI_2 + \begin{bmatrix} 1 & -1 \\ -1 & 2.5 \end{bmatrix}\right) = s^2+3.5s+1.5. \tag{47}$$

As both γ_i and λ_i are real in this example, the poles and zeros must lie on the root-locus curve for $M(s) = (s+1)/s^2$, as shown in Fig. 7. The minimal controllable subspace is by Theorem 11 equal to five ($\delta_{14}+1 = 4+1$), hence, the system is controllable from the node 1.

The transfer function $T_{\text{wz},13}(s)$ from the input disturbance $d_{\text{in}1}$ of agent 1 to the output y_3 is

$$T_{\text{in},13}(s) = \frac{y_3(s)}{d_{\text{in}1}(s)} = 0.3\frac{1}{s}\frac{(s+1)^2\prod_{i=1}^{2}(s^2+\gamma_i s+\gamma_i)}{\prod_{i=2}^{5}(s^2+\lambda_i s+\lambda_i)}. \tag{48}$$

The structure is the same as predicted in Theorem 7, since $M_s(s) = G(s) = 1/s$. The network part remains unchanged.

VIII. CONCLUSION

In this paper we considered transfer functions between two nodes in an arbitrary formation of identical SISO agents with an output coupling. Using the algebraic properties of forests in the graph, both numerator and denominator of the transfer function were derived in a simple form of a product of closed-loop polynomials with non-unit feedback gain. The transfer function for general input and output consists of two parts: the feedback part (fixed for a given pair of nodes) and the open-loop part.

The gains in the denominator and numerator polynomials are the roots of polynomials in the single-integrator system. If there is only one path between the controlling and observing nodes, the numerator gains are given as eigenvalues of the principal submatrix of the Laplacian. Finally, it is shown that the minimal dimension of the controllable subspace grows with the maximal distance from the controlling node.

Although it is hard to tell any transient properties from the location of poles and zeros — there are simply too many of them — still the product form can serve as an analytical tool. For instance, it may help in the analysis of the scaling in distributed control designs. We have already applied some of the results in [4] to the analysis of the scaling of the $\mathcal{H}_\infty$ norm, where the underlying topology was a path graph.

APPENDIX A
PROOF OF THEOREM 3

Before the proof, we need the following technical lemma.

Lemma 12. *Let $(L^k)_{oc}$ be the o, c element of L^k. Then*

$$\sum_{i=1}^{N} \rho_i v_{oi} \lambda_i^k = (L^k)_{oc}, \tag{49}$$

Proof. Since $\rho_i = e_i^T V^{-1} e_c$ and $v_{oi} = e_o^T V e_i$, we get

$$\begin{aligned}\sum_{i=1}^{N} v_{o,i} \lambda_i^k \rho_i &= \sum_{i=1}^{N} e_o^T V e_i \lambda_i^k e_i^T V^{-1} e_c \\ &= e_o^T V \left(\sum_{i=1}^{N} e_i \lambda_i^k e_i^T \right) V^{-1} e_c = e_o^T V \Lambda^k V^{-1} e_c \\ &= e_o^T L^k e_c = (L^k)_{oc}.\end{aligned} \tag{50}$$

This holds also for Jordan blocks in Λ larger than one. □

Proof of Theorem 3. Let us denote the numerator of the open loop in (7) as $\phi(s) = b(s)q(s)$ and the denominator as $\psi(s) = a(s)p(s)$. Note that the development here shows the case with simple Jordan blocks, although the proof remains valid for the case with larger blocks. The transfer function $T_{co}(s)$ can be obtained from (16) by using a common denominator as

$$T_{co}(s) = \frac{n(s)}{d(s)} = \sum_{i=1}^{N} \rho_i v_{oi} \frac{b(s)q(s)}{a(s)p(s) + \lambda_i b(s)q(s)} \tag{51}$$

$$\begin{aligned} &= \frac{\sum_{i=1}^{N} \left(\rho_i v_{oi}\, \phi(s) \prod_{j=1, j\neq i}^{N} [\psi(s) + \lambda_j \phi(s)] \right)}{\prod_{i=1}^{N} [\psi(s) + \lambda_i \phi(s)]} \\ &= \frac{\sum_{i=1}^{N} \rho_i v_{oi}\, \tau_i(s)}{\prod_{i=1}^{N} [\psi(s) + \lambda_i \phi(s)]},\end{aligned} \tag{52}$$

with $\tau_i(s) = \phi(s) \prod_{j=1, j\neq i}^{N} [\psi(s) + \lambda_j \phi(s)]$. Note that the polynomials in single-integrator dynamics are $h(s), g(s)$, while in higher-order dynamics they are $n(s), d(s)$. The denominator of (52) is the denominator in Theorem 3.

Having the denominator, we have to find the numerator $n(s)$. The polynomial $\tau_i(s)$ in (52) can be expanded in terms of powers of ϕ and ψ as (argument (s) is omitted)

$$\begin{aligned}\tau_i = \phi \prod_{j=1, j\neq i}^{N} [\psi + \lambda_j \phi] &= \psi^{N-1}\phi + \psi^{N-2}\phi^2 \left[\sum_{j=1, j\neq i}^{N} \lambda_j \right] \\ &+ \psi^{N-3}\phi^3 \left[\sum_{j=1, k=1, k\neq i\neq j}^{N} \lambda_j \lambda_k \right] + \ldots + \\ &+ \psi^1 \phi^{N-1} \left[\sum_{j=1, j\neq i}^{N} \left(\prod_{k=1, k\neq i\neq j}^{N} \lambda_k \right) \right] + \phi^N \left[\prod_{j=1, j\neq i}^{N} \lambda_j \right].\end{aligned} \tag{53}$$

Let us denote the the coefficients at the terms $\psi^j \phi^{N-j}$ in $\tau_i(s)$ as $\bar{\tau}_i^j$. They are given as a sum of all products of $N - j - 1$ eigenvalues. Then the polynomial $\tau_i(s)$ can be written as

$$\tau_i = \psi^{N-1}\phi + \bar{\tau}_i^{N-2} \psi^{N-2} \phi^2 + \ldots + \bar{\tau}_i^1 \psi \phi^{N-1} + \bar{\tau}_i^0 \phi^N. \tag{54}$$

The coefficients $\bar{\tau}_i^j$ can be simplified. Let us start with

$$\bar{\tau}_i^{N-2} = \sum_{j=1, j\neq i}^{N} \lambda_j = g_{N-1} - \lambda_i, \tag{55}$$

since the coefficient g_{N-1} of $g(s)$ is by (19) $g_{N-1} = \sum_{i=1}^{N} \lambda_i$. Similarly, the second coefficient is using (19)

$$\bar{\tau}_i^{N-3} = \sum_{j=1, k=1, k\neq i\neq j}^{N} \lambda_j \lambda_k = g_{N-2} - \lambda_i (g_{N-1} - \lambda_i). \tag{56}$$

For the last coefficient we get

$$\bar{\tau}_i^0 = \prod_{j=1, j\neq i}^{N} \lambda_j = g_1 - \lambda_i (g_2 - \lambda_i (g_3 - \lambda_i (\ldots))). \tag{57}$$

Knowing the coefficients $\bar{\tau}_j^i$, the numerator polynomial $n(s)$ can be using (52) written as

$$\begin{aligned} n(s) = \sum_{i=1}^{N} \rho_i v_{oi} \tau_i(s) &= \psi^{N-1}\phi \left(\sum_{i=1}^{N} \rho_i v_{oi} \right) \\ &+ \psi^{N-2}\phi^2 \left(\sum_{i=1}^{N} \rho_i v_{oi}\, \bar{\tau}_i^{N-2} \right) + \ldots \\ &+ \psi \phi^{N-1} \left(\sum_{i=1}^{N} \rho_i v_{oi}\, \bar{\tau}_i^1 \right) + \phi^N \left(\sum_{i=1}^{N} \rho_i v_{oi}\, \bar{\tau}_i^0 \right).\end{aligned} \tag{58}$$

The coefficients $\bar{h}_i$ of individual powers of $\psi^i \phi^{N-i}$ in $n(s)$ can be simplified using Lemma 12 and the formulas for $\bar{\tau}_j^i$ (55-57). The first two read

$$\bar{h}_{N-1} = \sum_{i=1}^{N} \rho_i v_{oi} = (L^0)_{oc} \tag{59}$$

$$\begin{aligned}\bar{h}_{N-2} = \sum_{i=1}^{N} \rho_i v_{oi}\, \bar{\tau}_i^{N-2} &= g_{N-1} \left(\sum_{i=1}^{N} \rho_i v_{oi} \right) - \sum_{i=1}^{N} \rho_i v_{oi} \lambda_i \\ &= g_{N-1} (L^0)_{oc} - (L^1)_{oc}\end{aligned} \tag{60}$$

Using the same ideas, the other coefficients $\bar{h}_i$ are

$$\bar{h}_{N-3} = g_{N-2} (L^0)_{oc} - g_{N-1} (L^1)_{oc} + (L^2)_{oc} \tag{61}$$

$$\vdots$$

$$\bar{h}_0 = g_1 (L^0)_{oc} - g_2 (L^1)_{oc} + \ldots + (L^{N-1})_{oc}. \tag{62}$$

The general form is now apparent,

$$\bar{h}_i = \sum_{j=0}^{N-i-1} g_{i+j+1} (-L)^j_{oc}. \tag{63}$$

Using the coefficients $\bar{h}_i$ in (59)-(62), the numerator $n(s)$ in (58) equals

$$\begin{aligned} n(s) = \phi(s) \Big(&\bar{h}_{N-1} \psi(s)^{N-1} + \bar{h}_{N-2} \psi^{N-2}(s) \phi(s) \\ &+ \bar{h}_{N-3} \psi^{N-3}(s) \phi^2(s) + \ldots + \bar{h}_0 \phi^{N-1}(s) \Big).\end{aligned} \tag{64}$$

Now we show that the coefficients $\bar{h}_i$ in (64) are equal to the coefficients h_i of the numerator polynomial $h(s)$ in the single integrator dynamics, i.e., $\bar{h}_i = h_i, \forall\, i$.

To see this, Corollary 4 in [25] gives us a relation

$$\mathrm{adj}(sI + L) = \sum_{k=0}^{N-1} \left(\sum_{j=0}^{N-k-1} g_i s^{N-j-1} \right) (-L^k / s^k). \quad (65)$$

The coefficient matrix Γ_i at s^i in (65) is then defined as

$$\Gamma_i = \sum_{j=0}^{N-i-1} g_{i+j+1} (-L)^j. \quad (66)$$

Taking as an element of interest the o, cth element in $\mathrm{adj}(sI + L)$, we see by (63) that the coefficients $(\Gamma_i)_{oc} = \bar{h}_i$. Moreover, since by (21) $\mathrm{adj}(sI + L)_{oc}$ is equal to the numerator polynomial in single-integrator dynamics, we get $h_i = \bar{h}_i$, $\forall i$.

All the coefficients h_i are functions of the powers of the Laplacian. Using Lemma 1, it is clear that $h_i = 0$ for $i > N - \delta_{co}$, since all $(L^j)_{oc}$ for $j = 0, 1, \ldots, \delta_{co} - 1$ are zeros. Then in (20), $N_n = N - \delta_{co} - 1$ also for directed weighted graphs. This result allows us to rewrite (64) as

$$\begin{aligned} n(s) = \phi^{1+\delta_{co}}(s) \Big(& h_{N-\delta_{co}-1} \psi^{N-1-\delta_{co}}(s) \quad (67) \\ & + h_{N-\delta_{co}-2} \psi^{N-2-\delta_{co}}(s) \phi(s) + \ldots + h_0 \phi^{N-1-\delta_{co}}(s) \Big). \end{aligned}$$

Previous equation can be factored into a product

$$n(s) = h_{N-\delta_{co}-1} \phi^{1+\delta_{co}}(s) \prod_{i=1}^{N-1-\delta_{co}} \Big(\psi(s) + \gamma_i \phi(s) \Big), \quad (68)$$

where the scalars $-\gamma_i$ are the roots of the polynomial $h(s)$ defined in (20). They are thus the zeros of the transfer function for the single integrator dynamics.

Note that $\bar{h}_{N-\delta_{co}-1} = h_{N-\delta_{co}-1} = \vartheta_{co}$ by Lemma 2. Then we get the numerator as

$$n(s) = \vartheta_{co}\, \phi^{1+\delta_{co}}(s) \prod_{i=1}^{N-1-\delta_{co}} \Big(a(s)p(s) + \gamma_i b(s) q(s) \Big). \quad (69)$$

Now in (69) and (52) we have both the numerator $n(s)$ and the denominator $d(s)$ of (23), which concludes the proof. $\square$

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