Stability of a circular system with multiple asymmetric Laplacians

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Abstract: We consider an asymptotic stability of a circular system where the coupling Laplacians are different for each state used for synchronization. It is shown that there must be a symmetric coupling in the output state to guarantee the stability for agents with two integrators in the open loop. Systems with agents having three or more integrators cannot be stabilized by any coupling. In addition, recent works in analysis of a scaling in vehicular platoons relate the asymptotic stability of a circular system to a string stability. Therefore, as confirmed by simulations in the paper, our results have an application also in path graphs.

Keywords: Multiple Laplacians, circular system, stability, vehicular platoon, two integrators

1. INTRODUCTION

Multi-agent coordination and synchronization is now a field of intensive research. The majority of works today use the same interconnection (same graph Laplacian) for all states used for coupling. On the other hand, some others suggest that by using different Laplacians a better transient can be achieved.

For systems where the individual subsystems (agents, vehicles, . . .) use the same interaction topology for all states we have a lot of useful results for stability, performance and even optimal control. Since these systems use only one Laplacian, a block diagonalization approach similar to that in a paper by Fax and Murray (2004) helps a lot in an analysis and synthesis. The paper by Zhang et al. (2011) uses a Riccati-based design of a state-feedback controller to easily stabilize the system for various interconnections. The only parameter to be redesigned is the gain of the controller which is proportional to the second smallest eigenvalue of Laplacian. A similar approach appeared with a use of passifiability in a paper by Fradkov and Junussov (2011). Later on, it was proved by Movric and Lewis (2014) that Riccati-based design achieves inverse optimality. Arcak (2007) shows that for symmetric interactions of passive systems stability is guaranteed even for nonlinear systems.

The properties of a single graph Laplacian were used in the analysis of scaling in vehicular platoons. The paper Hao and Barooah (2012) shows that when asymmetry between front and rear spacing error is introduced, bound on eigenvalues can be achieved for arbitrary large formation. This would guarantee controllability and better transient

time. On the other hand, papers by Tangerman et al. (2012); Herman et al. (2015a) show that with such bound and asymmetry both in position and velocity, the scaling of $\mathcal{H}_\infty$ norm of the platoons is very bad. In fact, whenever there is more than one integrator in the open-loop, the $\mathcal{H}_\infty$ norm scales exponentially with the graph distance.

The motivation for this paper indeed comes out from the analysis of scaling in vehicular platoon. Recent works by Hao et al. (2012); Cantos et al. (2014); Herman et al. (2015b) show that if vehicles in a platoon use symmetry in a coupling in position and asymmetry in a coupling in velocity, a good transient response can be achieved. Such a system can be modelled using two Laplacian matrices. Since in general multiple Laplacian matrices are not simultaneously diagonalizable, analysis of the overall system of vehicles gets very difficult. This is true especially when we are interested in how the system performance scales with the number of agents or vehicles.

The systems with different coupling for each state were not much investigated in the literature, despite the fact that different Laplacians neither yield the control law more complicated, nor they require more communication. The only (but very important) limitation is a much more complicated analysis. In this paper we had chosen a simpler way — we based our development on the property of simultaneous diagonalizability of circulant matrices. Thanks to that we can derive necessary conditions on the communication coupling for general agents. Circular systems are often considered as a simplification of more complicated systems. Spatial invariance property was used in paper by Bamieh et al. (2012) to show scaling of effect of the noise. Here we are mainly interested in asymptotic stability for very large number of agents.

We work with identical SISO systems in controller canonical form. The papers by Cantos et al. (2014) and Herman et al. (2015b) both have two integrators in the agent's model and both have symmetric coupling in the output. Does this mean that symmetry in coupling is necessary for more complicated models? Our answer is affirmative: we show that in order to achieve asymptotic stability for two integrators in the vehicle model, the coupling in the output state must be symmetric, i. e. Laplacian of an undirected circular graph must be used. The coupling in other states can be asymmetric. If there are more than two integrators in the open loop, the system can never be stabilized for a number of vehicles high enough.

The paper Cantos and Veerman (2014) states a reasonable conjecture that when a system, whose interconnection can be modelled as a circular graph, is asymptotically unstable, then the system with an underlying path-graph topology is either asymptotically unstable or flock unstable. Flock instability means that the response of the system scales exponentially in the number of agents. Therefore, if the conjecture holds true, our results are applicable to the analysis of scaling even in the path graph.

The paper is structured as follows. In the next section we describe the system model. In the third section the necessary conditions for asymptotic stability are shown. Next the results are illustrated using simulations of circular and path graphs.

2. SYSTEM MODEL

We consider N identical agents which exchange information about their states over a communication graph with a circular topology. The coupling can be asymmetric and each state can use different asymmetry.

All agents have identical SISO models of higher order — the order n of the agent can be arbitrary. We assume that the agent is modelled in a controller canonical form

$$\dot{x}_i = A x_i + B u_i \qquad (1)$$

with matrices $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times 1}$. The state vector of an individual agent is given as $x_i = [x_{i,0}, x_{i,1}, \ldots, x_{i,n-1}]^T$ and the matrices as

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -p_0 & -p_1 & -p_2 & \ldots & -p_{n-1} \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \qquad (2)$$

The characteristic polynomial of the open loop is clearly

$$p(s) = s^n + p_{n-1} s^{n-1} + p_{n-2} s^{n-2} + p_{n-3} s^{n-3} + \ldots + p_0 \quad (3)$$

We assume there are r integrators in the open loop, therefore $p_i = 0$ for $i = 0, \ldots, r-1$. The most common cases are one, two or three integrators. We will call the state $x_{i,0}$ as the output state.

Remark 1. The agent's model can be a combination of a plant model and a dynamic controller. We assume in this case that (1) models an open loop of the system — controller in series with plant — composed together. Although in principle the agents can exchange all states, the dynamic controller may still be necessary to satisfy the internal model principle, see papers by Wieland et al. (2011); Lunze (2012). For instance, in vehicular formations with a nearest-neighbor interaction, two integrators in the

open loop are necessary for leader tracking (see a paper by Barooah and Hespanha (2005)).

The communication (or measurement) is used to exchange the information about states of neighboring agents. Each state can use different asymmetry of the interaction. We assume that $m \le n$ states are exchanged and those are the states $x_{i,0}, x_{i,1}, \ldots, x_{i,m-1}$ — the output and its $m-1$ derivatives. We will use index i to denote the index of the agent while index j is used to index the state of an individual agent. Thus, $x_{i,j}$ is the jth state of the ith agent.

Since we work with a circular communication topology, the Laplacians describing the interconnections are given as the following circulant matrices

$$L_j = \begin{bmatrix} 1 & -\rho_j & 0 & 0 & \ldots & -(1-\rho_j) \\ -(1-\rho_j) & 1 & -\rho_j & 0 & \ldots & 0 \\ 0 & -(1-\rho_j) & 1 & -\rho_j & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\rho_j & 0 & \ldots & 0 & -(1-\rho_j) & 1 \end{bmatrix}$$
$$(4)$$

with $j = 0, 1, \ldots, m-1$ and ρ_j is the asymmetry of coupling of the jth state. If $\rho_j = 0.5$, then the coupling of the state is symmetric. The eigenvalues of the Laplacian L_j are given as (Cantos and Veerman (2014))

$$\lambda_j(\phi) = [1 - \cos\phi + \imath(1 - 2\rho_j)\sin\phi] \qquad (5)$$

with $\phi = \frac{2\pi k}{N}$, $k = 0, 1, \ldots, N-1$ and $\imath = \sqrt{-1}$. Since we are interested in a behavior of formations with large number of agents, we will treat the eigenvalues as a continuous function of $\phi \in [0, 2\pi]$. The eigenvalues of L_j are complex unless $\rho_j = 0.5$.

The control law for each agent is given as

$$u_i = \sum_{j=0}^{m-1} g_j(1-\rho_j)(x_{i-1,j} - x_{i,j}) - \rho_j g_j(x_{i,j} - x_{i+1,j}). \quad (6)$$

The terms g_j are the coupling gains for the state j. The control law is a weighted error to the neighbor's states. Using matrices it is

$$u_i = C[(I - \Omega)(x_{i-1} - x_i) - \Omega(x_i - x_{i+1})] \qquad (7)$$

with $C = [g_0, g_1, \ldots, g_{n-1}]$, I is the identity matrix and $\Omega = \text{diag}(\rho_0, \rho_1, \ldots, \rho_{n-1})$, $g_j = \rho_j = 0$ for $j \ge m$. Equation (7) is a neighbor's static-state feedback.

After the coupling of all the states is incorporated, the overall state space model has a form

$$\dot{x} = A_c x + B_c u \qquad (8)$$

The state vector is given as a stacked vector $x = [x_{1,0}, x_{2,0}, \ldots x_{N,0}, x_{1,1}, x_{2,1}, \ldots, x_{N,1}, \ldots, x_{1,n}, x_{2,n}, \ldots, x_{N,n}]^T$, that is, first are the states $x_{i,0}$ for all N vehicles, then the states $x_{i,1}$ for all vehicles, etc. The matrices are

$$A_c = \begin{bmatrix} 0 & I & 0 & \ldots & 0 \\ 0 & 0 & I & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\hat{P}_0 & -\hat{P}_1 & -\hat{P}_2 & \ldots & -\hat{P}_{n-1} \end{bmatrix}, B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ I \end{bmatrix} \quad (9)$$

We introduced the matrices $\hat{P}_j = I p_j + g_j L_j$. If we do not want to use coupling at jth state, we can set $g_j = 0$, therefore $g_j = 0$ for $j \ge m$.

3. STABILITY ANALYSIS

We calculate the eigenvalues ν of A_c as $A_c w = \nu w$. In vector form with $w = [w_0^T, w_1^T, \ldots, w_n^T]^T$,

$$\begin{bmatrix} 0 & I & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\hat{P}_0 & -\hat{P}_1 & \ldots & -\hat{P}_{n-1} \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ \ldots \\ w_{n-1} \end{bmatrix} = \nu \begin{bmatrix} w_0 \\ w_1 \\ \ldots \\ w_{n-1} \end{bmatrix} \quad (10)$$

It follows that $w_j = \nu w_{j-1}$. The last row gives us

$$-\hat{P}_0 w_0 - \hat{P}_1 w_0 \nu - \ldots - \hat{P}_{n-1} w_0 \nu^{n-1} = w_0 \nu^n \quad (11)$$

If follows that w_0 must be an eigenvector of all matrices $\hat{P}_j$, from which we get that it is also an eigenvector of L_j. All circular matrices are simultaneously diagonoalizable by discrete Fourier transform, that is, the eigenvectors w_j of L_j have elements

$$(w_j)_k = e^{\imath \phi k}. \quad (12)$$

Hence,

$$\hat{P}_j w_0 = (I p_j + g_j L_j) w_0 = (p_j + g_j \lambda_j(\phi)) w_j. \quad (13)$$

From (11) we can get N characteristic equations of A_c of the form

$$\nu^n + (p_{n-1} + g_{n-1} \lambda_{n-1}(\phi)) \nu^{n-1} \quad (14)$$
$$+ \ldots + (p_1 + g_1 \lambda_1(\phi)) \nu + p_0 + g_0 \lambda_0(\phi) = 0.$$

Plugging the eigenvalues λ_j from (5), we get

$$\nu^n + (p_{n-1} + g_{n-1} [1 - \cos\phi + \imath(1 - 2\rho_{n-1}) \sin\phi]) \nu^{n-1}$$
$$+ \ldots + (p_1 + g_1 [1 - \cos\phi + \imath(1 - 2\rho_1) \sin\phi]) \nu \quad (15)$$
$$+ p_0 + g_0 [1 - \cos\phi + \imath(1 - 2\rho_0) \sin\phi] = 0.$$

This is a complex-coefficient characteristic polynomial.

Let $\beta_j = 1 - 2\rho_j$. We can expand the eigenvalues (5) of Laplacian L_j in a Taylor series around $\phi = 0$ as

$$\lambda_j(\phi) = \left[\imath \beta_j \phi + \frac{1}{2} \phi^2 - \frac{\imath}{6} \beta_j \phi^3 \ldots \right]. \quad (16)$$

3.1 Conditions on interconnection

In this paper we analyze the conditions for stability of the matrix A_c in (8) when the number of vehicles gets very high, $N \to \infty$. We are not interested in finding the stability conditions for one particular model of the vehicle. Instead, we would like to see what are the requirements on the communication topology.

Let us first analyze the necessary conditions for stability when the polynomial (15) is only real.

Lemma 2. A necessary condition for asymptotic stability of (8) is that the following real-coefficient polynomials

$$\nu^n + (p_{n-1} + 2g_{n-1}) \nu^{n-1} + \ldots + (p_1 + 2g_1) \nu + p_0 + 2g_0 = 0. \quad (17)$$

and

$$\nu^n + p_{n-1} + \ldots + p_1 \nu + p_0 = 0. \quad (18)$$

are stable.

Proof. Set $\phi = \pi$ in (15) to get (17) and $\phi = 0$ to get (18). $\square$

Since the roots of (18) must be in the left half-plane, all the curves $\nu(\phi)$, $\phi \in [0, 2\pi]$ start in the left-half plane. The only exception are the poles at origin (there are r poles at the origin) in (18). In order to guarantee stability of A_c, the curves must not leave closed left half-plane. That is, they must not cross the imaginary axis.

The following theorem is the main result of the paper. It shows that symmetric coupling in the output state $x_{i,0}$ is a necessary condition for stability when there are two integrators in the open loop. Moreover, systems with three integrators cannot be stabilized for N large enough.

Theorem 3. Let r be the number of integrators in the agent model (2). Then as $N \to \infty$, the system (8)

(1) is unstable if $r = 2$ and $\rho_0 \neq 0.5$.
(2) is unstable if $r > 2$.

If $r = 1$ the stability depends on the system and tuning of the parameters.

Proof. Since by Lemma 2 $n - r$ roots of (18) lie in the open left half-plane and r of them are at the origin, we will be interested in the behavior close to the origin. Let us investigate what are the roots of (15) as $\phi \to 0$. Since ϕ is small, we can keep only two lowest order terms (to keep both real and imaginary parts) of the Taylor expansion in (16), i.e., $\lambda_j(\phi) \approx \imath \beta_j \phi + \frac{1}{2} \phi^2$. The polynomial (15) is then

$$q(\nu) = \nu^n + (p_{n-1} + g_{n-1} [i\beta_{n-1}\phi + \frac{1}{2}\phi^2]) \nu^{n-1}$$
$$+ \ldots + (p_1 + g_1 [i\beta_1\phi + \frac{1}{2}\phi^2]) \nu \quad (19)$$
$$+ p_0 + g_0 [i\beta_0\phi + \frac{1}{2}\phi^2] = 0.$$

We can decompose it to three polynomials as

$$q(\nu) = q_1(\nu) + \frac{1}{2}\phi^2 q_2(\nu) + \imath\phi q_3(\nu) \quad (20)$$

where the polynomials are defined as

$$q_1 = p(\nu) = \nu^n + p_{n-1}\nu^{n-1} + \ldots + p_1\nu_1 + p_0 \quad (21)$$
$$q_2 = g_{n-1}\nu^{n-1} + g_{n-2}\nu^{n-2} + \ldots g_1\nu + g_0 \quad (22)$$
$$q_3 = g_{n-1}\beta_{n-1}\nu^{n-1} + g_{n-2}\beta_{n-2}\nu^{n-2} + \ldots + g_0\beta_0. \quad (23)$$

We can convert the complex-coefficient polynomial q in (19) to the real-coefficient polynomial $\hat{q}$ by

$$\hat{q}(\nu) = q(\nu)\bar{q}(\nu), \quad (24)$$

where $\bar{q}$ has all coefficients as complex conjugates of those in q. Then we can write

$$\hat{q}(\nu) = \left[q_1(\nu) + \frac{1}{2}\phi^2 q_2(\nu) + \imath\phi q_3(\nu) \right] \left[q_1(\nu) + \frac{1}{2}\phi^2 q_2(\nu) \right.$$
$$\left. -\imath\phi q_3(\nu) \right] = \left[q_1(\nu) + \frac{1}{2}\phi^2 q_2(\nu) \right]^2 + [\phi q_3(\nu)]^2$$
$$= q_1^2(\nu) + \phi^2 q_1(\nu) q_2(\nu) + \frac{1}{4}\phi^4 q_2^2(\nu) + \phi^2 q_3^2(\nu). \quad (25)$$

The polynomial $\hat{q}(\nu)$ is stable if and only if $q(\nu)$ is stable. Since ϕ is small, the terms with ϕ^4 can be neglected in (25). Then equation (25) has a form

$$\hat{q}(\nu) \approx q_1^2(\nu) + \phi^2 \left[q_1(\nu)q_2(\nu) + q_3^2(\nu) \right] \quad (26)$$

This can be viewed a closed-loop polynomial of the system $M(\nu)$ defined as

$$M(\nu) = \phi^2 \frac{q_1(\nu)q_2(\nu) + q_3^2(\nu)}{q_1^2(\nu)}. \quad (27)$$

The term ϕ^2 acts as a gain in the closed loop. The closed-loop system $\phi^2 M(\nu)/(1 + \phi^2 M(\nu))$ is stable if and only if $\hat{q}(\nu)$ is stable for all ϕ^2. Recall that there are r integrators

in the open loop of the system and $p_i = 0$ for $i = 0, \ldots, r-1$. Then ν^r can be factored out from q_1 to get $q_1 = \nu^r \hat{q}_1$, where $\hat{q}_1$ has a nonzero absolute term. $M(\nu)$ then reads

$$M(\nu) = \phi^2 \frac{\nu^r \hat{q}_1(\nu) q_2(\nu) + q_3^2(\nu)}{\nu^{2r} \hat{q}_1^2(\nu)}. \quad (28)$$

Such a system has $2r$ poles at the origin. Using the root-locus rules (Dorf and Bishop, 2008, p. 418), when we close the loop these poles will start to move on the trajectories in a complex plane separated by angles $2\pi/(2r)$. Therefore, if there are more than 2 poles at the origin, at least one branch will go to the right half-plane. Thus, the system will be unstable for small ϕ. In order to cancel the unwanted poles at the origin and keep at most two of them, we require that there are at least $2r-2$ zeros at the origin in the numerator of $M(\nu)$.

If $r = 2$, then we need two zeros at the origin. That is, we must be able to factor ν^2 out of the numerator of (28). Such term is already present in $\nu^r \hat{q}_1(\nu) q_2(\nu)$. To assure that ν^2 can also be factored out of $q_3^2(\nu)$, we require that the absolute term in $q_3(\nu)$ is zero. This is achieved from (23) by setting $\beta_0 = 0$, which means $\rho_0 = 0.5$. Then we obtain two zeros at the origin, as required. If the output state uses nonsymmetric coupling, we cannot have two zeros at the origin and the system (8) is unstable. The root-locus for one particular system is shown in Fig. 1a for symmetric output state and in Fig. 1b for asymmetric output state.

If $r > 2$ we require that there are $2r-2$ zeros at the origin. Since the numerator is $\nu^r \hat{p}(\nu) q_2(\nu) + q_3^2(\nu)$, we can affect only the lowest r coefficients of the numerator to be zero ($\nu^r \hat{p}$ has the lowest order r and cannot be affected by the interconnection). But for $r > 2$ we have that $r < 2r-2$ and we cannot have sufficient number of zeros. The system (8) is therefore unstable. $\square$

The stability conclusion is the same as for the symmetric circular system, where we also cannot have more than two integrators (Barooah and Hespanha (2005)). The symmetric coupling in the output state holds for all models and all orders of the system. Thus, the results of Cantos and Veerman (2014); Herman et al. (2015b) with symmetric coupling in positions are special cases of this theorem.

If the necessary conditions of Lemma 2 and Theorem 3 are satisfied, then we can use the imaginary axis as a guardian map.

Lemma 4. The system (8) is asymptotically stable if the following equation

$$\begin{aligned}
(\imath\omega)^n &+ (p_{n-1} + g_{n-1}[1 - \cos\phi + \imath(1 - 2\rho_{n-1})\sin\phi])(\imath\omega)^{n-1} \\
&+ \ldots + (p_1 + g_1[1 - \cos\phi + \imath(1 - 2\rho_1)\sin\phi])(\imath\omega) \quad (29) \\
&+ p_0 + g_0[1 - \cos\phi + \imath(1 - 2\rho_0)\sin\phi] = 0.
\end{aligned}$$

has no solution for all $\omega \in \mathbb{R}$.

Proof. As discussed above, by Lemma 2 all curves $\nu(\phi)$ start in the left half plane or at the origin. Suppose that the necessary conditions following from Theorem 3 are satisfied. When the curves $\nu(\phi)$ do not cross the imaginary axis, the system (8) is stable. Therefore, there must not exist a solution to (15) which has purely imaginary roots. That is why when $\imath\omega$ is plugged for ν to (15), it must not have a solution. This fact is captured by (29). $\square$

4. SIMULATIONS

In this section we verify our results numerically for a particular system. Our example is inspired by vehicular platoons. The individual system is a vehicle which should track its neighbors in a formation. Suppose the model of the vehicle is a double integrator with a viscous friction (velocity feedback), given in a transfer function as $G(s) = \frac{1}{s^2 + 0.5s}$. Its output is the vehicle's position y. For such a model we designed a controller $R(s) = \frac{14.3s^2 + 14.3s + 3}{s^2 + 3s}$. The controller connected in series with the vehicle model form the open-loop $M(s) = R(s)G(s)$. The open loop has 2 integrators, hence it satisfies Internal Model Principle for tracking of a ramp signal, caused by the platoon's leader moving with a constant velocity.

The open loop can be modelled in a controller-canonical form as

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1.5 & -3.5 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, C = [3 \ 14.3 \ 14.3 \ 0] \quad (30)$$

The state vector is $[y, v, c_1, c_2]$ with the states being position, velocity, controller state 1 and controller state 2, respectively. We will use the terms in the matrix C as a coupling coefficients, i. e., $g_0 = 3, g_1 = 14.3$ and $g_2 = 14.3$. That is, position, velocity and output controller state are used for control. Note that at least a coupling in the position and velocity is a necessary stability condition. To see this, consider the characteristic polynomials in (15). The coefficients p_0 and p_1 are zero due to the open loop model. If we also set g_0 and g_1 equal to zero, (15) would have two lowest coefficients missing, implying instability.

The overall system has a form

$$A_c = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -3L_y & -14.3L_v & -14.3L_c - 1.5I & -3.5I \end{bmatrix}, B_c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix} \quad (31)$$

The Laplacians have a form of (4) with asymmetries ρ_y, ρ_v, ρ_c for L_y, L_v, L_c, respectively. We will vary the asymmetries to illustrate the stability and instability. The characteristic equation (15) has now a form

$$\nu^4 + 3.5\nu^3 + (1.5 + 14.3\lambda_c(\phi))\nu^2 + 14.3\lambda_v(\phi)\nu + 3\lambda_y(\phi) = 0 \quad (32)$$

First we illustrate the root-locus approach used in the proof of Theorem 3. The transfer function is $M(s) = \frac{q_1(s)q_2(s) + q_3^2(s)}{q_1^2(s)}$. In this case, the polynomials would be $q_1 = s^4 + 3.5s^3 + 1.5s^2, q_2 = 14.3s^2 + 14.3s + 3$ and $q_3 = (1 - 2\rho_c)14.3s^2 + (1 - 2\rho_v)14.3s + (1 - 2\rho_y)3$. The figure 1 shows a plot of root-locus of $M(s)$ for: a) $\rho_y = 0.5, \rho_v = 0.45, \rho_c = 0.35$ (symmetry in position) and b) $\rho_y = 0.48, \rho_v = 0.45, \rho_c = 0.35$ (small asymmetry in position). It is clear that for the asymmetric position the roots lie in the right half-plane, so the system (31) gets unstable. When there is a symmetry in the position and an asymmetry in the other states, stability is achieved.

The Fig. 2 illustrates that the eigenvalues ν_i of the second-order Taylor series approximation (19) match those calculated using exact formula (15) and also those obtained as
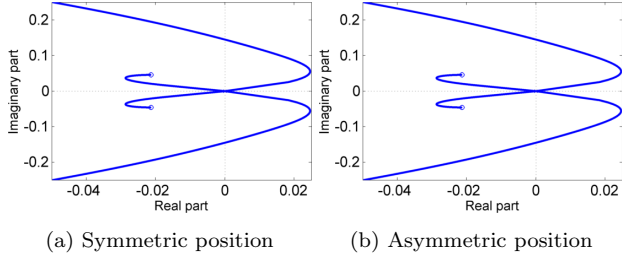
(a) Symmetric position

(b) Asymmetric position

Fig. 1. Root locus plots for $M(s)$ from the proof of Theorem 3.

(a) Symmetric position: $\rho_y = 0.5, \rho_v = 0.45, \rho_c = 0.35$

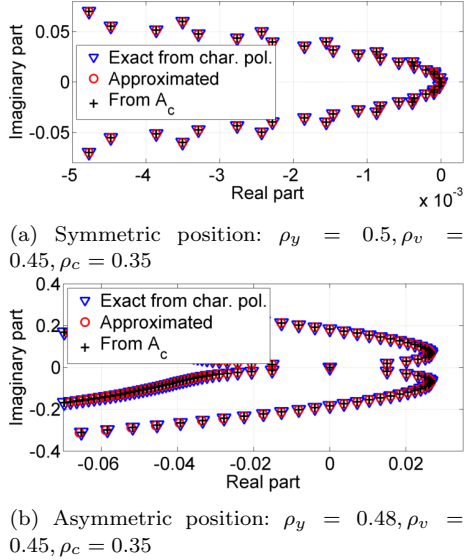(b) Asymmetric position: $\rho_y = 0.48, \rho_v = 0.45, \rho_c = 0.35$

Fig. 2. Eigenvalue locations for $N = 1000$ calculated using different formulas. ∇ - calculation based on (15), o - calculation based on (19) and $+$ shows eigenvalues of A_c. Note different scales of axes.
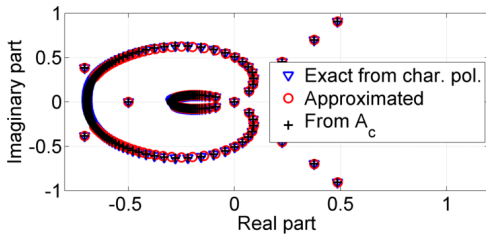

Fig. 3. Eigenvalues for $N = 100$ for a system with three integrators (33). Legend is the same as in the Fig. 2.

the eigenvalues of A_c. The figure also confirms Theorem 3, since the system with the asymmetric coupling in the position is asymptotically unstable — the eigenvalues are in the right half-plane.

For three integrators in the open loop it is impossible to design an interaction achieving asymptotic stability. Let us show it using the following model

$$A_c = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -3L_y & -14.3L_v & -14.3L_c & -3.5I \end{bmatrix}, B_c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix} \tag{33}$$

with $\rho_y = 0.5, \rho_v = 0.5$ and $\rho_c = 0.3$, that is, the only asymmetry is in the controller state. The eigenvalues are shown in Fig. 3. As expected, such a system is unstable.
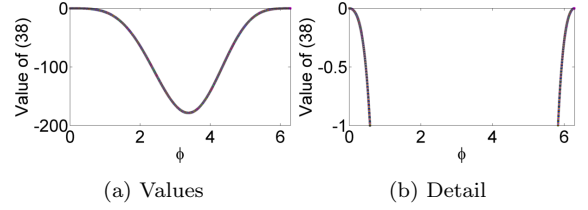

(a) Values

(b) Detail

Fig. 4. The value of polynomial in (36) for negative solutions of (38). For the positive solution the values are in the same range. For every ϕ the solution (38) was plugged to (36) for the same value of $\phi \in [0, 2\pi]$.

4.1 Test of asymptotic stability

Let us test if the system (31) is asymptotically stable for all N. The analysis here is just for illustration, it should not give any general results. We will use the Lemma 4. Plugging the values for asymmetry $\rho_y = 0.5, \rho_v = 0.45, \rho_c = 0.35$ and for eigenvalues from (5) to (32) gives

$$\nu^4 + 3.5\nu^3 + (1.5 + 14.3[1 - \cos\phi + \imath 0.3\sin\phi])\nu^2 + 14.3[1 - \cos\phi + \imath 0.1\sin\phi]\nu + 3[1 - \cos\phi + 0\imath\sin\phi] = 0 \tag{34}$$

Plugging $\imath\omega$ for ν and testing the crossing of imaginary axis as in Lemma 4, the characteristic equation gets a form

$$\omega^4 - 3.5\imath\omega^3 - (1.5 + 14.3[1 - \cos\phi + \imath 0.3\sin\phi])\omega^2 + 14.3[1 - \cos\phi + \imath 0.1\sin\phi]\imath\omega + 3[1 - \cos\phi + 0\imath\sin\phi] = 0 \tag{35}$$

The real part of the equation (35) is

$$\omega^4 - (1.5 + 14.3[1 - \cos\phi])\omega^2 - 14.3 \cdot 0.1\sin(\phi)\omega + 3[1 - \cos\phi] = 0 \tag{36}$$

and the imaginary part can be factored as

$$\omega(-3.5\omega^2 - 4.29\sin(\phi)\omega + 14.3[1 - \cos\phi]) = 0. \tag{37}$$

The solution $\omega = 0$ is not a nontrivial (for $\phi \neq 0$) solution of (36). The solutions $\omega_1(\phi), \omega_2(\phi)$ of (37) are

$$\omega_{1,2}(\phi) = \frac{4.29\sin\phi \pm \sqrt{4.29^2\sin^2\phi + 200.2(1 - \cos\phi)}}{-14}. \tag{38}$$

The equation (36) was numerically evaluated for all $\omega_{1,2}$ calculated for a range of $\phi \in [0, 2\pi]$. The resulting curve is in Fig. 4. It shows that zero is not a value of (36) unless $\phi = 0$ which gives the polynomial (18) with stable roots or roots at the origin. This confirms that (34) has no nontrivial solution, hence the system is stable for all N.

4.2 Relation to a path graph

As we stated in the introduction, the motivation for circular system analysis stems from behavior of a vehicular platoon. In this section we will experimentally confirm the conjecture of Cantos and Veerman (2014) that if the circular system is asymptotically unstable, the path system will be asymptotically or flock unstable. The thorough explanation of this conjecture can be found in the papers by Cantos and Veerman (2014) or Herman et al. (2015b). The main idea is that for vehicles far away from the boundaries the behavior of the two systems is very similar. Therefore, if in the circular graph the propagating signal is amplified, it will do the same in the path graph as well.

(a) Symmetric all states: $\rho_y = \rho_v = \rho_c = 0.5$

(b) Asymmetric all states: $\rho_y = \rho_v = \rho_c = 0.45$

(c) Symmetric position, asymmetric all other states: $\rho_y = 0.5, \rho_v = 0.45, \rho_c = 0.35$

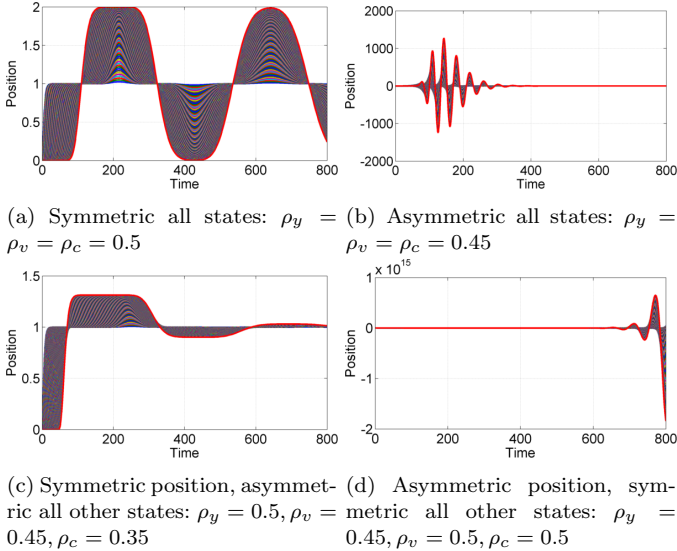(d) Asymmetric position, symmetric all other states: $\rho_y = 0.45, \rho_v = 0.5, \rho_c = 0.5$

Fig. 5. Comparison of different asymmetries in the system with a path graph — a vehicular platoon with $N = 150$. Note different scales in the y-axis.

The comparison of different asymmetries is shown in Fig. 5. It shows the response of the formation to the step in leader's position. It is clear that completely symmetric interaction (Fig. 5a) has a very long transient. Contrary, when the coupling in all states is asymmetric (Fig. 5b), the transient has a very high overshoot — the flock instability appears. The symmetric coupling in the position and the asymmetric coupling in the other states (Fig. 5c) shows very good transient with low overshoot. Asymmetry in position and symmetry in other states in Fig. 5d results even in asymptotic instability of the path system. The simulations therefore confirm the conjecture of Cantos and Veerman (2014) even for our fourth-order system. It also suggests that a partial asymmetry can substantially improve the transient.

Thus, it seems that the property of necessity of symmetric coupling in position is valid in general for systems with two integrators in the open loop. This would extend the results of papers by Hao et al. (2012); Cantos and Veerman (2014); Herman et al. (2015b) to more general models. However, a general proof is still missing.

5. CONCLUSION

In this paper we analyzed the asymptotic stability of an interconnection of agents of arbitrary order with underlying circular topology. Different and asymmetric Laplacians were used. We proved that it is necessary to have a symmetric interaction for the output state for agents having two integrators in the open loop. A system of agents having three integrators cannot be stabilized for a number of agents large enough. Our simulations also confirm the conjecture of Cantos and Veerman (2014) that asymptotic instability of a circular system means flock instability of a path system.

For a future work, we would like to generalize the property of good transients in path-graph systems with a partial asymmetry to arbitrary systems. Also the ideas for optimization of the transient time from Herman et al. (2015b) perhaps can be extended to general systems.

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